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# Anyon condensation

Topological symmetry breaking phase transitions and  
commutative algebra objects in braided tensor categories

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## Abstract

This work has two aspects. We discuss topological order for planar systems and explore a graphical formalism to treat topological symmetry breaking phase transition. In the discussion of topological order, we focus on the role of quantum groups and modular tensor categories. Especially the graphical formalism based on category theory is treated extensively. We have incorporated topological symmetry breaking phase transitions, induced by a bosonic condensate, in this formalism. This allows us to calculate general operators for the broken phase using the data for the original phase. As an application, we show how to treat topological symmetry breaking on the level of the topological  $S$ -matrix and illustrate this in two representative examples from the  $\mathfrak{su}(2)_k$  series. This approach can be viewed as an application of the theory of commutative algebra objects in modular tensor categories.

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*Sometimes a scream is better than a thesis*

Manfred Eigen

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# CHAPTER 1

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## Introduction

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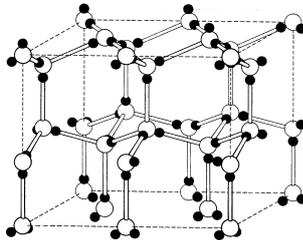
The theory of phases and phase transitions plays a central role in our understanding of many physical systems. Topological phases, or topologically ordered phases, pose an interesting problem for theorists in this respect, as they fall outside of the conventional scheme to understand phases that occur in terms of the breaking of symmetries. The formalism of topological symmetry breaking, based on the breaking of an underlying quantum group symmetry, can be viewed as an extension of this theory to the context of topological phases. We will discuss topological symmetry breaking and connect it to the notion of commutative algebras in modular tensor categories. In particular, we put topological symmetry breaking in a graphical form, applying notions from tensor categories, which gives the freedom to calculate general operators for the phases under consideration.

In condensed matter physics, a fundamental problem is the determination of the low-temperature phases or orders of a system. Dating back to Lev Landau [52, 53], the theory of symmetry breaking phase transitions forms a corner stone in this respect. Put simply, there are two aspects to the picture it provides: symmetry breaking and particle condensation. This mechanism underlies interesting phenomena, such as superconductivity – where the electric  $U(1)$  symmetry is broken by a condensate of Cooper pairs – but also the formation of ice (figure 1.1). Group theory provides the right language to discuss many aspects of symmetry breaking. The classification of the different phases is essentially equivalent to the classification of subgroups of the relevant symmetry group.

In the past few decades, a growing interest has emerged to study topological phases that are not the product of (group) symmetry breaking. Especially in two dimensions, topological phases offer access to fundamentally new physics. They can have anyonic quasi-particle excitations – particles that are neither bosons nor fermions. These offer a route to the fault-tolerant storage and manipulation of quantum information known as topological quantum computation (TQC), which may some day be used to realize

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**Figure 1.1: The structure of an ice crystal** - If water freezes, or any other liquid-to-solid transition, the molecules order in a regular lattice structure. This breaks the translational and rotational symmetry of the fluid state. Using group theory, one can see that there are 230 qualitatively different types of crystals corresponding to the 230 space groups

the dream of building a quantum computer.

The best known physical realization of topological phases occurs in the fractional quantum Hall fluids. These exotic states in two-dimensional electron gases submitted to a strong perpendicular magnetic field have quantum numbers that are conserved due to topological properties, not because of symmetry. Similar phases might occur in rotating Bose gases, high  $T_c$  superconductors, and possibly many more systems. One can in fact show that there is an infinite number of different topological phases possible, which suggests a world of possibilities if we ever gain enough control to engineer systems that realize a phase of choice.

From a mathematical point of view, interesting structures have entered the theory. The description and study of topological phases involve conformal field theory, topological quantum field theory, quantum groups and tensor categories, and all these structures are heavily interlinked. They are studied by mathematicians and theoretical physicists alike and link topics like string theory, low-dimensional topology and knot theory to condensed matter physics.

In this thesis we discuss topological ordered planar systems, and in particular the description using quantum groups and how these lead to tensor categories. In fact, we prefer the tensor category viewpoint from which the theory can be neatly summarized by a set of  $F$ -symbols and  $R$ -symbols. We show how to calculate these in examples related to discrete gauge theories and Chern-Simons theory.

Our main goal is to discuss how the breaking of quantum group symmetry can be understood from the perspective of tensor categories. In the topological symmetry breaking scheme, a key role is played by the formation of a bosonic condensate. In this thesis, we argue that this notion corresponds to a commutative algebra object in braided tensor categories.

Using the formulation of topological symmetry breaking in terms of tensor categories, one can write down and calculate diagrammatic expressions for operators in the broken phase. This allows us to describe phase transition on the level of the topological  $S$ -matrix, an important invariant for the theory. We illustrate this in two representative examples. This shows the usefulness of the topological  $S$ -matrix, related to the

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exchange statistics of the particles, as an indicator for topological orders.

Another important indicator for topological order is the topological entanglement entropy. The quantum dimension of the condensate, or quantum embedding index, appears as a universal quantity characterizing the phase transitions and relating the topological entanglement entropy of the different phases.

This thesis is organized as follows.

## Chapter 2

In this chapter, we discuss some aspects of topological order in greater detail, namely anyons and the fractional quantum Hall effect. The larger picture of how conformal field theory, topological field theory and topological order are related is discussed briefly. We also point out some other developments in the field.

## Chapter 3

The appearance of quantum groups as the underlying symmetry for topological phases is discussed. Discrete gauge theories are taken as a representative example. The underlying symmetry is the so-called quantum double  $D[H]$ . It can be understood physically as an algebra of gauge transformations and flux projections, which gives an intuitive picture of how interacting electric and magnetic degrees of freedom can lead to materialized quantum group symmetry. On a more formal level, we discuss the quantum group  $U_q[\mathfrak{su}(2)]$  which is important for the quantization of Chern-Simons theory and  $\mathfrak{su}(2)_k$  Wess-Zumino-Witten models.

## Chapter 4

Here, we introduce in detail the graphical formalism for topological phases. One needs a finite set of charges, a description of fusion rules, so called  $F$ -symbols and  $R$ -symbols to describe the topological properties of anyons completely. An important quantity is the topological  $S$ -matrix. Together with the  $T$ -matrix it forms a representation of the modular group. As an application and illustration of the graphical formalism we give a proof of the Verlinde formula. Finally, quantum states, density matrices and the calculation of physical amplitudes is discussed.

## Chapter 5

The calculation of topological data for the quantum double  $D[H]$  is illustrated and we present general formulas for the  $\mathfrak{su}(2)_k$  models.

## Chapter 6

In this chapter, we start the discussion of topological symmetry breaking. Topological symmetry breaking is a way to construct from a theory  $\mathcal{A}$  describing the unbroken phase, a theory  $\mathcal{T}$  and a theory  $\mathcal{U}$ . The  $\mathcal{T}$ -theory describes the broken phase, but may

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include excitations that braid non-trivially with the condensate. These excitations cannot occur in the bulk of the broken phase as the disruption of the condensate costs energy. They get confined to the boundary or as  $\mathcal{T}$ -hadrons. Projecting out the confined excitations one gets the unconfined theory  $\mathcal{U}$ -that describes the bulk of the broken phase.

We put this scheme in the context of the graphical formalism of chapter 4. We discuss how to find a well-defined condensate and how to construct the particle spectrum upon this. The conditions for confinement are reconsidered and it is shown that there is an operator that projects out confined excitations.

### Chapter 7

The quantum dimension of the condensate – or quantum embedding index –  $q$  appears as a universal ratio between the quantum dimensions of the broken phase and the unbroken phase. This can physically be understood as a relation in terms of the topological entanglement entropy. We show how this universality of  $q$  follows from the assumption of topological symmetry breaking.

Finally, we give the main application of the graphical formalism for topological symmetry breaking. We show that the symmetry breaking phase transition can be described directly on the level of the topological  $S$ -matrix. The diagrammatic calculation gives insight in the nature of topological symmetry breaking phase transitions. The condensate appears explicitly where in the original formulation it could always be left out.

### Chapter 8

The last chapter contains concluding remarks and and indicates possible paths for future research.

In this chapter, we give a brief discussion of some essential topics concerning topological phases and topological order in 2+1 dimensions. These are anyons, the FQH effect, some generalities of the mathematical structures that are involved. This chapter serves as more elaborate introduction and to sketch a bit of history of the topic. Relevant references are included.

## 2.1 Anyons

In the most mundane spacetime, that of 3+1 dimensions, particles fall in precisely two classes: bosons and fermions. These can either be defined by their exchange properties or by their spins. The importance of this difference between particles can of course not be overstated. It is at the root of the Pauli exclusion principle and Bose-Einstein condensation, and determines whether macroscopic numbers of particles obey Fermi-Dirac or Bose-Einstein statistics.

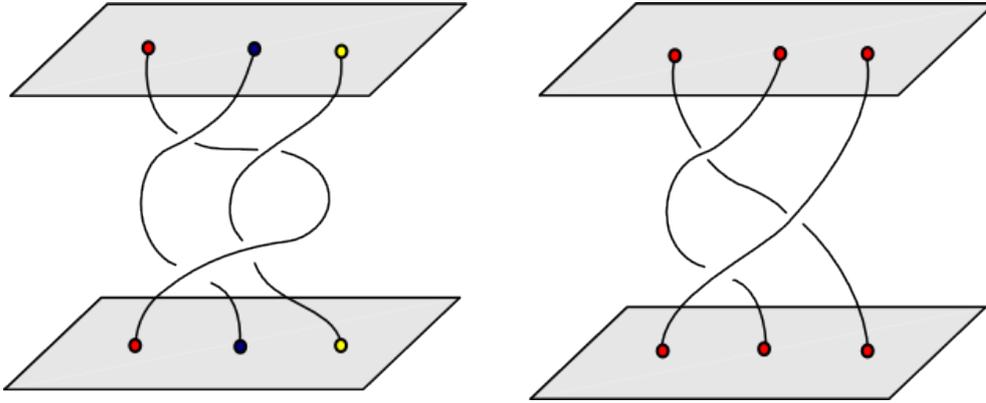
According to textbook quantum mechanics, the wave function of a multi-particle state of fermions is obtained by anti-symmetrising the tensor product of single-particle wave functions, while for bosons the wave function is symmetrised. This procedure long stood as a corner stone of quantum mechanics and is still procedure in many applications. But one may object that in a quantum theory with indistinguishable particles, the labelling of particle coordinates has no physical significance and brings unobservable elements into the theory.

The consequences of above observation were first fully appreciated by Leinaas and Myrheim. In their seminal paper [57], they show that the exchange properties of particles are tightly connected to the topology of the configuration space of multi-particle systems. In fact, as was shown in [51], particle types correspond to unitary representations of the fundamental group of this space.

## 2. TOPOLOGICAL ORDER IN THE PLANE

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Let us discuss this in a spacetime picture. The trajectories of particles can now conveniently be imagined by their world lines through spacetime. We will consider trajectories that leave the system in a configuration identical to the initial configuration and that do not have intersecting world lines. These correspond to free propagation of the system without scattering events. If the particles are distinguishable, the world lines should start and end at the same spatial coordinates. If the particles are indistinguishable, the world lines are allowed to interchange spatial coordinates.



**Figure 2.1: Spacetime trajectories** - Two spacetime trajectories for a system of three particles with identical initial and final states. If the particles are of different type, world lines should start and end at the same coordinates in space (left), while for indistinguishable particles, permutations of spatial coordinates are allowed (right).

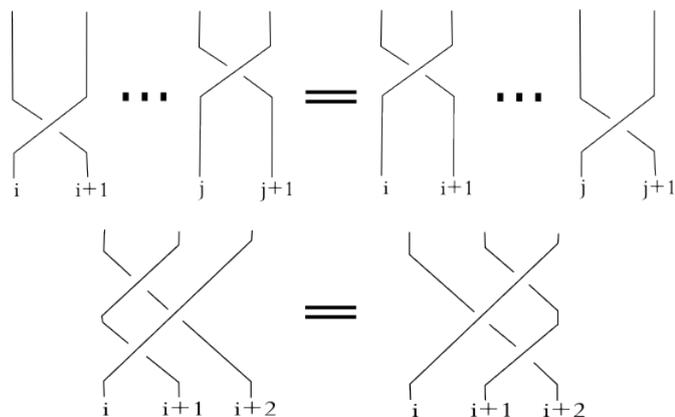
The trajectories fall in distinct topological classes, depending on whether or not they can be deformed into each other by smooth deformations of the world lines (without intermediate intersections). If space has three or more dimensions one can see that there is no topologically different notion of ‘under’ and ‘over’ crossing of world lines, but any exchange of coordinates is simply that: an exchange. Therefore the exchange of particles is governed by the permutation group  $S_N$ .

However, when the particles live in a plane – so spacetime is three-dimensional – such a distinction of crossings becomes essential to incorporate the difference between interchanging the particles clockwise or counter-clockwise. The relevant group is not  $S_N$  but  $B_N$ , Artins  $N$ -stranded braid group [3].

This group is generated by the two different interchanges or half-braidings

$$R_i = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ i \qquad i+1 \end{array}, \quad R_i^{-1} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ i \qquad i+1 \end{array} \quad (2.1)$$

which are inverse to each other. Here,  $1 \leq i \leq N - 1$  labels the strand of the braid. These generators are subject to defining relations corresponding to topological manipulations of the braid. Pictorially they are given by the following equations.



We can also write them algebraically as

$$\begin{aligned}
 R_i R_j &= R_j R_i && \text{for } |i - j| \geq 2 \\
 R_i R_{i+1} R_i &= R_{i+1} R_i R_{i+1} && \text{for } 1 \leq i \leq N - 1
 \end{aligned}
 \tag{2.2}$$

The second relation is the famous Yang-Baxter equation [15, 86, 87].

As remarked before, the important difference between particles in the plane and particles in three-dimensional space is the difference between over and under crossings of world lines. The relation to statistics in higher dimensions represented on the level of groups due the fact that we can pass from the braid group  $B_N$  to the permutation group  $S_N$  by implying the relation  $R_i = R_i^{-1}$ , or equivalently  $(R_i)^2 = 1$ . This one extra relation makes a huge difference for the properties of the group. While the permutation group is finite ( $|S_N| = N!$ ), the braid group is infinite, even for two strands. The representation theory of the braid group is therefore much richer than that of the permutation group. This gives the possibility of highly non-trivial exchange statistics in 2+1 dimensions, also referred to as 'braid statistics'. Particles that obey these exotic braid statistics were dubbed *anyons* by Wilczek [80]. Bosons and fermions, in this context, just correspond to two of an infinite number of possibilities. Since distinguishable anyons can also have nontrivial braiding, it is better to think of braiding statistics as a kind of topological interaction.

Bosons and fermions correspond to the two one-dimensional representations of  $S_N$ , even and odd respectively. In principle, higher dimensional representations of  $S_N$ , known as 'parastatistics' [39], could lead to more particle types, but it has been shown that these can be reduced to the one-dimensional representations at the cost of introducing additional quantum numbers [31].

One-dimensional representations of the braid group simply correspond to a choice of phase  $\exp(i\theta)$  assigned to an elementary interchange. Since, in this case the order of the exchange is unimportant because phases commute, anyons corresponding to one-dimensional representations are called Abelian. However, it is no longer true that higher dimensional representations of the braid group can be reduced. This gives rise to so called non-Abelian anyons, particles that implement higher dimensional braid

## 2. TOPOLOGICAL ORDER IN THE PLANE

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statistics. These non-Abelian anyons are the ones that can be used for topological quantum computation as suggested by Kitaev in [49]. See [64, 66] for a review.

### 2.2 The fractional quantum Hall effect

Because the most prominent system with anyonic excitations is the fractional Quantum Hall effect, we include a brief discussion of some relevant aspects. For greater detail, we refer to the literature. See e.g. [65, 88] for a thorough introduction.

In 1879, Edwin Herbert Hall discovered a phenomenon in electrical conductors that we know today as the *Hall effect*. Under influence of an applied electric field, the electrons will start moving in the direction of the field, which leads to a current. But the application of a perpendicular magnetic field will, due to the Lorentz force, lead to a component to the current which is orthogonal to both the electric and magnetic field. This is the Hall effect.

As an idealization, we may consider a two-dimensional electron gas (2DEG) with coordinates  $(x, y) = (x^1, x^2)$  that we submit to an electric field in the  $x$ -direction and a perpendicular magnetic field  $B$  in the  $z$  direction. The relation between the current and the electric field is given by the resistivity tensor  $\rho$  and the conductivity  $\sigma$ , through the relations

$$E^\mu = \rho_{\mu\nu} J^\nu, \quad J^\mu = \sigma_{\mu\nu} E^\nu \quad (2.3)$$

Assuming homogeneity of the system, one can use a relativistic argument to deduce that [37]

$$\rho = \frac{B}{nec} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma = \frac{nec}{B} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (2.4)$$

with  $n$  the electron density and  $B$  the strength of the magnetic field. Remarkably, since  $\rho_{xx} = 0$  and  $\sigma_{xx} = 0$ , the system behaves as a perfect insulator and a perfect conductor in the direction of the electric field simultaneously.

The Hall resistivity is defined as  $\rho_H = \rho_{xy}$ . According to the equations above, we must have

$$\rho_H = \frac{B}{nec} \quad (2.5)$$

The derivation of this equation leans critically on Lorentz invariance, but not on much more. Therefore, if there is no preferred reference frame, this result is very robust. It should in particular hold whether we consider quantum or classical electrodynamics. It is therefore striking that it does not agree with experimental data. Experiments reveal that at low temperatures ( $\sim 10$  mK) and high magnetic field ( $\sim 10$  T) the dependence of  $\rho_H$  on  $B$  is not linear, but instead plateaus develop at precisely quantized values

$$\rho_H = \frac{1}{\nu} \frac{h}{e^2} \quad (2.6)$$

The number  $\nu$  is called the filling fraction and is usually written as  $\nu = N_e/N_\Phi$ . Here  $N_e$  is the number of electrons, while  $N_\Phi$  is the number of flux-quanta piercing the 2DEG. To get a better understanding of the filling fraction, let us elaborate on its meaning.

## 2.2 The fractional quantum Hall effect

From analysing the single-particle Hamiltonian, one sees that the energy spectrum forms so called Landau levels

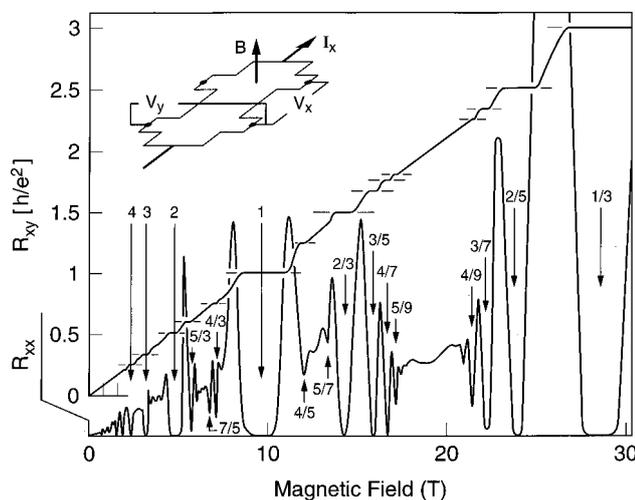
$$E_n = (n + \frac{1}{2})\hbar\omega_c, \quad \omega_c = \frac{eB}{mc} \quad (2.7)$$

In a finite sized system of area  $A$ , the available number of states for each Landau level turns out to be [37]

$$N_\Phi = A \frac{B}{\Phi_0} \quad (2.8)$$

Here  $\Phi_0 = \frac{hc}{e}$  is the fundamental flux quantum, so apparently the number of available states for each energy level is the same as the number of flux-quanta that pierce the system. Hence,  $\nu = N_e/N_\Phi$  is the ratio between the number of filled energy states – the electrons – and the number of available energy states, which is why it is called the filling fraction.

Plateaus at integer values for the filling fraction were first reported in 1980 by von Klitzing et al. [50], and are referred to as the integer quantum Hall effect (IQHE). Plateaus at fractional filling fractions – found two years later by Tsui et al. [74] – are the hallmark for the fractional quantum Hall effect (FQHE). While the IQHE can essentially be understood from the single particle physics, neglecting the Coulomb interaction, the FQHE represents an intriguing interplay of many interacting electrons. The Coulomb interaction is crucial to explain its features.



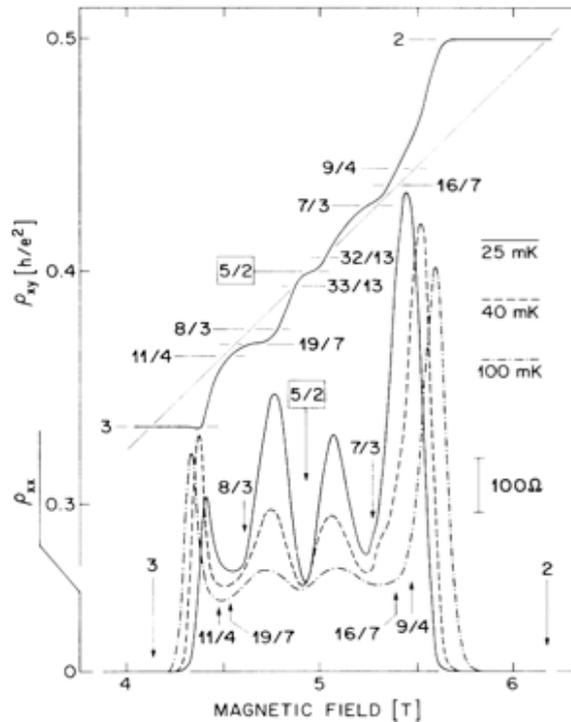
**Figure 2.2: The quantum Hall effect** - The Hall resistance  $R_{xy} = R_H = V_y/I_x$  and the longitudinal resistance  $R_{xx} = V_x/I_x$  are plotted against a varying magnetic field. (In two dimensions, the resistance is the same as the resistivity:  $R = \sigma$ .) An illustration of the measurement set-up is also given on the top left. Taken from ref. [73]

Figure 2.2 shows a typical graph of the plateaus in the first Landau level, the  $\nu = 1/3$  being the plateau that was first discovered. Note that the longitudinal resistivity  $\sigma_{xx}$

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drops to zero at the plateaus. which means that the conductivity tensor is off-diagonal as in equation (2.4). Hence, a dissipationless transverse current flows in response to an applied electric field. In particular, when we thread an extra magnetic flux quantum through the medium, the system will expel a net charge of  $\nu e$  due to the induced magnetic field. Or, put differently, this will create a quasi-hole of charge  $-\nu e$  and one flux quantum, illustrating the intimate coupling between charge and flux in the quantum Hall effect. It is a first indication that these systems harvest elementary quasi-particle excitations that are charge-flux composites, which, due to the internal Aharonov-Bohm interaction, can be anyons. The peculiar fact that these quasi-particles have fractional charge when  $\nu$  is non-integer has been experimentally verified in [38].

With the application of topological quantum computing in mind, most interest lies in systems exhibiting non-Abelian anyons. In this respect, a series of plateaus that has been observed in the second Landau level [81] offers the most promising platform. Especially the  $\nu = \frac{5}{2}$  plateau has received much attention. See figure 2.3.



**Figure 2.3: Plateaus in the second Landau level** - The resistivity is plotted against a varying magnetic field. The graph shows plateaus in the second Landau level. The filling fraction  $\nu = p/q$  with  $2 < \nu < 4$  is indicated. Taken from ref. [81].

We will leave most of the theoretical subtleties concerning the quantum Hall effects untouched. In the paragraph below, however, we do want to point out the relation of conformal field theory (CFT) and topological quantum field theory to FQH wave functions, as it relates to our discussion of topological symmetry breaking phase transitions.

### 2.2.1 Conformal and topological quantum field theory

Over the years, a series of wave functions has been proposed to capture the physics of the fractional quantum Hall effect at different plateaus, the Laughlin wave function [55] being the first. It can account for the plateaus at filling fraction  $\nu = 1/M$  for odd  $M$ , and reads

$$\Psi_L(z) = \prod_{i < j} (z_i - z_j)^M \exp \left[ \frac{1}{4} \sum_i |z_i|^2 \right] \quad (2.9)$$

Here, the  $z_i$  label the complex coordinates for  $N_e$  electrons.<sup>1</sup>

The Laughlin state describes an electronic system where the electrons carry magnetic vortices. This attachment of vortices to electrons is paradigmatic for the various quantum Hall states that have been proposed in the literature. Note that to ensure a fermionic system, i.e. the wave function is anti-symmetric under interchange of the coordinates, we need  $M$  to be odd.

Skipping subsequent developments in the history of the FQHE (e.g. the Haldane-Halperin hierarchy [43, 44, 56] and the composite fermion approach by Jain [45]), we want to turn our attention to an observation first made by Moore and Read. In [63] they noted that the Laughlin wave function could be obtained as a correlator in a certain rational conformal field theory (RCFT). In fact, they argue that this relation should be general and conjecture that every FQHE state should be related to conformal field theory, which would give a means to classify FQHE states.

Without going into the details of conformal field theory (see e.g. [33, 36] for introductory texts and [62, 63] for further information), let us reflect for a moment on the picture they sketch. They motivate their search for a formulation in terms of RCFT by pointing to the development of Ginzberg-Landau effective field theories for the FQHE effect where the action contains a Chern-Simons term. Witten showed [83] that there is a connection between Chern-Simon theory in a three dimensional bulk and a Wess-Zumino-Witten theory on the two-dimensional edge. Moore and Read argue that the long range effective field theory should essentially constitute a pure Chern-Simons theory and that a related RCFT should come in to play when we consider a two-dimensional edge of the spacetime. Interestingly, the RCFT can account for wave functions by means of correlators in a (2+0)d interpretation, but also governs the edge excitation in a (1+1)d interpretation.

Chern-Simons theory is an example of a topological quantum field theory (TQFT). This is apparent from the fact that the spacetime metric does not enter the Chern-Simons action

$$S_{CS} = \frac{k}{4\pi} \int d^3x \operatorname{Tr} \epsilon^{\mu\nu\rho} (a_\mu \partial_\nu a_\rho + \frac{2}{3} a_\mu a_\nu a_\rho). \quad (2.10)$$

A topological quantum field theory is generally a quantum field theory for which all correlation functions are invariant under arbitrary smooth deformations of the base manifold [4, 84].

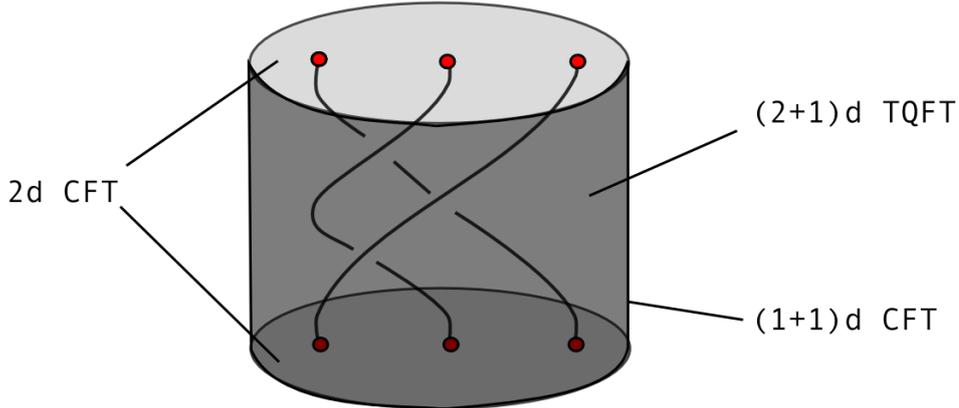
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<sup>1</sup>The magnetic length  $\ell_B = \sqrt{\frac{\hbar c}{eB}}$  is set to unity, or, equivalently, the complex coordinates are taken  $z = (x + iy)/\ell_B$ .

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The picture Moore and Read proposed for the FQHE is a general picture we have in mind when we discuss topological order. The long range physics is governed by an effective field theory that is a TQFT, hence we also speak of a topological phase. Often, the physics of a time slice and of gapless edge excitations is controlled by CFTs with the same topological order, although this might not be accurate in all situations.



**Figure 2.4: Holography** - A schematic view of how TQFT and CFT at least in certain theories, are related. A TQFT governs the bulk of the cylinder. Time slices are described by a (2+0) dimensional CFT. An example of this are the FQHE wave functions constructed from CFT correlators. Gapless boundary excitations are described by a CFT with the same topological order when there is no edge reconstruction. This can be regarded as an instance of the holographic principle.

Moore and Read proposed the following wave function that could explain an even denominator FQH plateau

$$\Psi_{\text{Pf}}(z) = \text{Pfaff} \left( \frac{1}{z_i - z_j} \right) \prod_{i < j}^M \exp \left[ \frac{1}{4} \sum_i |z_i|^2 \right] \quad (2.11)$$

where  $\text{Pfaff } A_{ij}$  is the square root of the determinant of  $A$ . The filling fraction for this state is  $\nu = 1/M$ . Its particular benefit lies in the fact that, for  $M = 2$ , it describes a fermionic system with  $\nu = 1/2$ . Fermionic systems with even denominators cannot be described by the Laughlin wave function and states derived from this. This Moore-Read state is a likely candidate for the plateau at  $\nu = 5/2 = 2 + \frac{1}{2}$ , which has two completely filled Landau levels and one level at half filling. As it features quasi-holes that are non-Abelian anyons and the  $\nu = 5/2$  plateau is experimentally accessible, this is a promising candidate for the experimental verification of non-Abelian braid statistics. This experimental challenge does not seem to have been met decisively yet.

The Moore-Read state is constructed from the so called Ising CFT. Subsequent generalization of the ideas of Moore and Read was done by Read and Rezayi [69]. They constructed a series of states based on CFT related to  $\mathfrak{su}(2)_k$  WZW models (that include the Ising CFT), that could explain plateaus in the second Landau level.

Interestingly, one of these states encompasses so called Fibonacci anyons which are universal for quantum computation.

### 2.3 Other approaches to topological order

The FQHE in 2DEGs is not the only place to look for (non-Abelian) anyons. Systems that might have states very similar to the FQH states discussed above are  $p_x + ip_y$  superconductors [26, 41] and rotating Bose gases [24, 25, 68], though the latter of course would display a bosonic version of the FQHE.

We should also note the existence of two lattice models that realize topological order, at this point. The first is known as the Kitaev model. In its most general form, this model features interacting spins placed on the edges of an arbitrary lattice [49]. The spins are labelled by the elements of a group. The Hamiltonian consists of mutually commuting projectors, and is therefore completely solvable. The projectors can be recognized as generalizations of either electric or magnetic interactions. The electric operators act on the edges joining at a vertex and ensure that the ground state transforms trivially under the action of the group, which can be seen as a sort of gauge transformation. The magnetic operators act on plaquettes and essentially project out states of trivial flux through the plaquette. In the simplest case, where the spins are labelled by  $\mathbb{Z}_2$  and a square lattice is taken, the ground state can be readily identified as a loop gas.

This lattice model realizes a discrete gauge theory.<sup>1</sup> For more information we refer to the literature. Especially [17] and [16] are interesting in the context of this thesis. They study the Kitaev model with boundary, which can be seen as an explicit symmetry breaking mechanism and relates to the phase transitions that we will discuss in subsequent chapters. These gauge theories might also be realizable in Josephson junction arrays [29].

The other paradigmatic lattice model for topological order was introduced by Levin and Wen under the name of string-net condensates [59]. This system lives on a trivalent lattice, where, again, spins are placed on edges and interact at vertices. The input is tensor category. We will discuss these structures in detail in chapter 4. The Hamiltonian has a similar structure as in the Kitaev model and is also completely solvable.

Remarkably, it has been shown that the Kitaev model can be mapped to a Levin-Wen type model, making the Kitaev model essentially a special case of the more general string-net models.

In the next chapter we will discuss the role of quantum groups in (2+1)d. In many concrete situations these can be identified as the underlying symmetry structure. Regardless of the name, these are actually not groups but should be regarded as a generalization. Formally, they are certain type of algebras with additional structures (see chapter 3 and appendix A).

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<sup>1</sup>Discrete gauge theories are discussed in chapter 3.

## 2. TOPOLOGICAL ORDER IN THE PLANE

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## CHAPTER 3

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### Quantum groups in planar physics

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In the previous chapter we discussed aspects of topological order, such as anyonic quasi-particle excitations and the quantum Hall effect. In the present chapter we will discuss the symmetries underlying topological order. Special to planar physics, especially when electric and magnetic degrees of freedom start to interact, is the occurrence of so called quantum group symmetry. Quantum groups generalize group symmetries in many ways. This full symmetry is often not manifest in the Hamiltonian or Lagrangian of the theory.

Symmetries of the Hamiltonian are represented on the quantum level as operators commuting with the Hamiltonian. Since commuting operators can be diagonalized simultaneously, the energy eigenstates can be labelled by quantum numbers coming from the symmetry operators. But it might happen that there is a larger set of commuting operators. This gives an idea of how symmetries not directly apparent from the Hamiltonian or Lagrangian can materialize. In this chapter we will first illustrate the occurrence of quantum group symmetry in the context of discrete gauge theories. The electric and magnetic excitations can be treated on equal footing, in these theories, when viewed as irreducible representation of the quantum double of the residual gauge group. Then we give a more formal treatment of the constituents of quantum groups, or quasi-triangular Hopf algebras as they are also called. Finally, we will discuss the quantum group  $U_q[\mathfrak{su}(2)]$  that is related to Chern-Simons theory and the Wess-Zumino-Witten model for conformal field theory.

A more formal treatment of the generalities of quantum groups is put forth the appendix [A](#).

#### 3.1 Discrete Gauge Theory

Suppose we start out with a (2+1)d Yang-Mills-Higgs theory with a gauge group  $G$ . We speak of a discrete gauge theory (DGT) when the continuous gauge group  $G$  is spontaneously broken down to a discrete, usually finite, subgroup  $H$ . Hence, this can be regarded as a gauge theory with residual gauge group  $H$ .

The Higgs mechanism causes the gauge field to acquire a mass, making all local interactions freeze out in the low-energy regime, effectively leaving only topological interactions. In this limit, the theory becomes a TQFT [11]. Below, we give a quick overview of the aspects that are important for our present purpose. See [67] for an elaborate exposition on which this approach is based.

A DGT has electric particles labelled by the irreps  $\{\alpha\}$  of the residual gauge group, which, before we mod out by gauge transformations to get the physical Hilbert space, carry the corresponding representation module  $V_\alpha$  as an internal Hilbert space. Furthermore, a DGT also has magnetic particles, or fluxes, associated to topological defects.<sup>1</sup> We think of the fluxes as defined by their effect on charges in an Aharonov-Bohm type scattering experiment [1]. An electric charge  $\alpha$  taken around a flux will feel the influence of the flux as a transformation in its internal Hilbert space  $V_\alpha$  by the action of some  $h \in H$ . We denote the flux as a ket  $|h\rangle$  labelled by the transformation  $h \in H$  it induces. However, since a flux measurement followed by a gauge transformation should give the same result as applying a gauge transformation first and do the flux measurement in the transformed system we find  $gh = h'g$ , where  $h, h'$  is the flux measured before or after the gauge  $g$  transformation respectively. Hence the gauge transformations has to act on the flux by conjugation, so the gauge invariant labelling is in fact by the conjugacy classes of  $H$  rather than just elements. Hence, while electric excitations carry representation modules as an internal Hilbert space, the internal space of fluxes is given by a conjugacy class.

Revealing the algebraic structure of the theory puts the foregoing on a firmer footing. Define the flux measurement operators  $\{P_h\}_{h \in H}$  that satisfy the projector algebra

$$P_h P_{h'} = \delta_{hh'} P_h \tag{3.1}$$

Since gauge transformations  $g \in H$  act on fluxes by conjugation, we must have

$$g P_h = P_{ghg^{-1}} g \tag{3.2}$$

The full algebra of gauge transformations and flux projections is spanned by the combinations

$$\{P_h g\}, \quad h, g \in H \tag{3.3}$$

that obey the multiplication rule

$$P_h g P_{h'} g' = \delta_{h,ghg^{-1}} P_h gg' \tag{3.4}$$

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<sup>1</sup>Note that we use electric and magnetic as general terminology for the two distinct types of particle-like excitations and it does not imply that the gauge group is  $U(1)$ .

This is in fact the quantum double  $D(H)$  of  $H$  which can be obtained from any finite group by a general construction due to Drinfel'd [30]. Note that the multiplication rule says that this algebra has a unit  $1 = \sum_h P_h e$ .

The full particle spectrum of a DGT with residual gauge group  $H$  can be recovered as the set of irreducible representations of  $D[H]$ .<sup>1</sup> Indeed, one finds back the irreps of  $H$  corresponding to electric particles, as well as the conjugacy classes, belonging to fluxes. But there is a third class of irreps corresponding to flux-charge composites or dyons. Physically, this is very interesting, because due to the Aharonov-Bohm effect, these composites can have fractional spin and can be true anyons.

The representation theory of  $D(H)$  was first worked out in [28], but we again follow [67]. It turns out that the irreducible representations of  $D(H)$  can be labelled uniquely by a conjugacy class  $A$  of  $H$ , and an irreducible representation of the *centralizer* of some representing element of  $A$ . Thus, when the conjugacy class  $\{e\}$  of the identity element is taken, or that of any other central element, we find the irreps of  $H$ . But apparently, for general non-trivial fluxes, not all transformations can be implemented straightforwardly on the charge part.

To make the action of  $D(H)$  explicit, we will have to make a few choices. Pick some order for the elements of  $A$  and write

$$A = \{ {}^A h_1, {}^A h_2, \dots, {}^A h_k \} \quad (3.5)$$

Let  $C(A)$  be the centralizer of  ${}^A h_1$  and fix a set  $X(A) = \{ {}^A x_1, {}^A x_2, \dots, {}^A x_k \}$  of representatives for the equivalence classes  $H/C(A)$ , such that  ${}^A h_i = {}^A x_i {}^A h_1 {}^A x_i^{-1}$ . Let us take  ${}^A x_1 = e$ , the unit element of  $H$ , for convenience. These choices will effect some of the specifics in what follows but a different choice would lead to a unitary equivalent representation of  $D(H)$ .

The internal Hilbert space  $V_\alpha^A$  of a particle with flux  $A$  and centralizer charge  $\alpha$  is now spanned by the quantum states

$$| {}^A h_i, {}^\alpha v_j \rangle, \quad \begin{array}{l} i = 1, \dots, k, \\ j = 1, \dots, \dim \alpha \end{array} \quad (3.6)$$

where we have chosen a basis  $\{ | {}^\alpha v_j \rangle \}$  for the centralizer representation  $\alpha$ . The action of an element  $P_h g$  of  $D(H)$  on these basis states, i.e. the effect of a gauge transformation  $g$  followed by a flux measurement  $P_h$ , is given by

$$\Pi_\alpha^A(P_h g) | {}^A h_i, {}^\alpha v_j \rangle = \delta_{h,g} \delta_{h_i g^{-1}} | g {}^A h_i g^{-1}, \Pi_\alpha(\tilde{g})_{mj} {}^\alpha v_m \rangle \quad (3.7)$$

(where summation over repeated indices is implied), with

$$\tilde{g} \equiv {}^A x^{-1} g {}^A x_i \quad (3.8)$$

and  ${}^A x_k$  the element of  $X(A)$  associated to  $h_k = g {}^A h_i g^{-1}$ . Note that the element  $\tilde{g}$  commutes with  ${}^A h_1$  as it should. Thus when acting on dyons,  $\tilde{g}$  is the part of the

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<sup>1</sup>We will assume representations to be unitary throughout the text

### 3. QUANTUM GROUPS IN PLANAR PHYSICS

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transformation that slips through the conjugation of the flux, and which is subsequently implemented on the charge.

This gives an explicit description of the irreps of  $D(H)$  including how the action of elements  $P_h g$  can be worked out. The single-particle states of the theory correspond to these irreps. Before we discuss multi-particle states, let us point out a subtlety concerning unitarity of the theory. In order to preserve the inner product of quantum states, we should have

$$\Pi_\alpha^A(P_h g)^\dagger = \Pi_\alpha^A(P_{g^{-1}hg} g^{-1}) \quad (3.9)$$

Hence it makes sense to define  $(P_h g)^* = P_{g^{-1}hg} g^{-1}$ , and more generally

$$\left( \sum_{h,g} c_{h,g} P_h g \right)^* = \sum_{h,g} c_{h,g}^* P_{g^{-1}hg} g^{-1} \quad (3.10)$$

where a  $*$  on the coefficients  $c_{h,g}$  just means complex conjugation. Formally, this turns  $D[H]$  into a Hopf- $*$ -algebra. Representations obeying (3.9) are said to respect  $*$ -structure. This gives the proper definition of a unitary representation.

#### 3.1.1 Multi-particle states

To let  $D(H)$  act on a two-particle state, one needs a prescription of how an element  $P_h g$  acts on  $V_\alpha^A \otimes V_\beta^B$ . Corresponding to intuitive expectations, the element  $P_h g$  implements the gauge transformation  $g$  on both particles and then projects out the total flux. Thus, we act on a state  $|^A h_i, {}^\alpha v_j\rangle |^B h_k, {}^\beta v_l\rangle$  with

$$\sum_{h'h''=h} \Pi_\alpha^A(P_{h'} g) \otimes \Pi_\beta^B(P_{h''} g) \quad (3.11)$$

Formally, this is nicely captured by the definition of a coproduct

$$\Delta: D(H) \rightarrow D(H) \otimes D(H) \quad (3.12)$$

by

$$\Delta(P_h g) = \sum_{h'h''=h} P_{h'} g \otimes P_{h''} g \quad (3.13)$$

The action of  $P_h g$  on a two-particle state is then defined as the application of  $\Pi_\alpha^A \otimes \Pi_\beta^B$  on  $\Delta(P_h g)$ , which indeed gives (3.11).

The action on three-particle states is produced by first applying  $\Delta$  to produce an element of  $D[H] \otimes D[H]$  and then apply  $\Delta$  again on either the left or the right tensor leg. Starting out with the element  $P_h g$ , either choice produces

$$\sum_{h'h''h'''=h} P_{h'} g \otimes P_{h''} g \otimes P_{h'''} g \quad (3.14)$$

as an element of  $D[H]^{\otimes 3}$ . Then we apply  $\Pi_\alpha^A \otimes \Pi_\beta^B \otimes \Pi_\gamma^C$  on this element to act on  $V_\alpha^A \otimes V_\beta^B \otimes V_\gamma^C$ . Note that when we describe the transformation in words, it is again

the application of the gauge transformation  $g$  on all three particles followed by a total flux measurement projecting out flux  $h$ .

Extending the action of  $P_h g$  on states with more than three particles is now straightforward. The resulting transformation is again the global gauge transformation  $g$  followed by the flux measurement. On a formal level, this is achieved by consecutive application of the coproduct to get an element in  $D[h]^n$  and letting each tensor leg act on the corresponding particle.

The tensor product representations of  $D(H)$  that we obtain using the coproduct are in general not irreducible, but they can always be decomposed into a direct sum of irreducible representations. Thus the space  $V_\alpha^A \otimes V_\beta^B$  can be written as a direct sum of subspaces that transform irreducibly, leading to the decomposition rules

$$\Pi_\alpha^A \otimes \Pi_\beta^B = \bigoplus_{(C,\gamma)} N_{\alpha\beta C}^{AB\gamma} \Pi_\gamma^C \quad (3.15)$$

These so-called fusion rules play an important role when we start discussing the general formalism for anyonic theories in the next chapter. Note that the representation  $\Pi^{\{e\}}_0$  has trivial fusion (where 0 denotes the trivial representation of  $H$ ). This corresponds to the zero-particle state or vacuum of the theory. The map  $\Pi^{\{e\}}_0: D[H] \rightarrow \mathbb{C}$  can also be called the counit, and is then often denoted with  $\epsilon$ . This is a general constituent of a quantum group, and should satisfy certain compatibility conditions with regard to the comultiplication.

### 3.1.2 Topological interactions

As remarked before, in the low-energy limit, the excitations of a DGT interact solely by Aharonov-Bohm type, topological interactions. A very nice feature of DGT theories is that the precise form of these interactions can be deduced from intuitive reasoning.

Suppose we have two fluxes with values  $h_1$  and  $h_2$ . As is illustrated in figure 3.1, the requirement that the total flux has to be conserved leads to the conclusion that a counter-clockwise interchange results in commutation of  $h_2$  by  $h_1$ . A charge crossing the  $h_1$  Dirac line that is depicted in the figure picks up the action of  $h_1$  in its internal Hilbert space. Hence we conclude that the general braiding operator acts as

$$\mathcal{R}_{(A,\alpha),(B,\beta)} \left( |^A h_i, \alpha v_j \rangle \left| ^B h_m, \beta v_n \right. \right) = \left| ^A h_i, \alpha v_j \right. \left. \left| ^B h_m, \beta v_n \right. \right) \left| ^A h_i, \alpha v_j \right. \left. \left| ^B h_m, \beta v_n \right. \right) \quad (3.16)$$

This transformation can be accomplished by composing the action of a special element in  $D[H] \otimes D[H]$ , called the universal  $R$ -matrix, on  $V_\alpha^A \otimes V_\beta^B$ , with flipping the tensor legs. The universal  $R$ -matrix for  $D[H]$  is

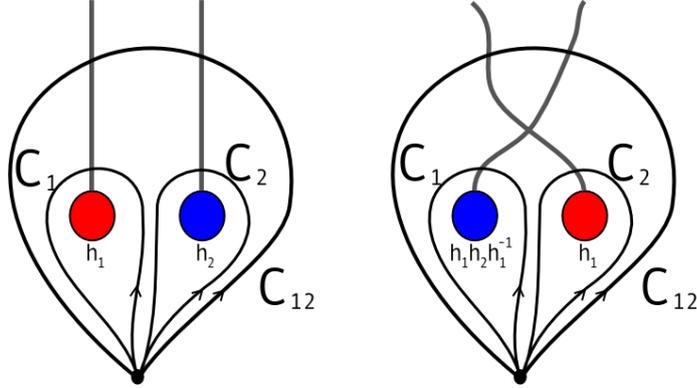
$$\mathcal{R} \equiv \sum_{g,h} P_g e \otimes P_h g \quad (3.17)$$

Acting on a two-particle state,  $\mathcal{R}$  indeed measures the flux of the left particle and implements it on the right particle. We can now define

$$\mathcal{R}_{(A,\alpha),(B,\beta)} = \tau(\Pi_\alpha^A \otimes \Pi_\beta^B)(\mathcal{R}) \quad (3.18)$$

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**Figure 3.1: The transformation of fluxes** - We start with two fluxes positioned next to each other in the plane, as depicted on the left. They carry flux  $h_1$  and  $h_2$  respectively. Fluxes are measured through Aharonov-Bohm interaction with test charges taken around paths  $C_1$ ,  $C_2$ . The total flux  $h_1 h_2$  as is measured using curve  $C_{12}$ . The grey lines show Dirac strings attached to the vortices. On the right hand side, the fluxes are interchanged counter-clockwise. Since the flux  $h_2$  crosses the Dirac string attached to  $h_1$ , its value can change to  $h'_2$ . Because the total flux is conserved we must have  $h'_2 h_1 = h_1 h_2$  hence we find that  $h'_2 = h_1 h_2 h_1^{-1}$ . So the half-braiding of fluxes leads to conjugation.

where  $\tau$  is the flip of tensor legs. It is easy to check that this gives (3.16) when applied on two-particle states.

The universal  $R$ -matrix has the following properties.

$$\mathcal{R}\Delta(P_h g) = \Delta(P_h g)\mathcal{R} \quad (3.19)$$

$$(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{23}\mathcal{R}_{12} \quad (3.20)$$

$$(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{23} \quad (3.21)$$

$$(3.22)$$

Here  $\mathcal{R}_{12} = \mathcal{R} \otimes 1$  and  $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$ . These consistency conditions ensure that the braiding is implemented consistently. The first relation tells us that the braiding commutes with the action of the quantum double on two-particle states, hence it acts as multiplication by a complex number on the irreducible subspaces (Schur's lemma). The other two conditions are known as the quasi-triangularity conditions. They imply the Yang-Baxter equation for the  $R$ -matrix,

$$\mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23} \quad (3.23)$$

On the level of representations, this leads to the Yang-Baxter equation for the braiding, thus implementing a representation of the braid group  $B_n$  on the  $n$ -particle Hilbert space of  $n$  identical particles.

To discuss the spin of general fluxes, charges, and dyons in this context, imagine an excitation  $(A, \alpha)$  as a composite where the flux  $A$  and the centralizer charge  $\alpha$  are kept apart by a tiny distance. The world line is now more like a ribbon with the flux

and the centralizer charge attached to opposite edges. A  $2\pi$  rotation corresponds to a full twist of the ribbon, such that the charge winds around the flux. This implements the flux on the charge. This is generally accomplished by the letting the element

$$\sum_h P_h h \quad (3.24)$$

act, which results in

$$\sum_h \Pi_\alpha^A(P_h h) |^A h_i, \alpha v_j\rangle = |^A h_i, \Pi_\alpha(h_1) \alpha v_j\rangle \quad (3.25)$$

as can be seen from working out the rule (3.7). Since the element  $h_1$  commutes by definition with all elements of the centralizer  $C(A)$ , we have

$$\Pi_\alpha(h_1) = e^{2\pi i h_{(A,\alpha)}} \mathbf{1}_\alpha \quad (3.26)$$

as this follows from Schur's lemma applied on the unitary irrep  $\alpha$ . The phase  $\theta_{(A,\alpha)} = \exp(2\pi i h_{(A,\alpha)})$  is the *spin* of the  $(A, \alpha)$ -particle. The element  $\sum_h P_h h$  from equation (3.24) is called the *ribbon element* of  $D[H]$ . Note that only true dyons can have  $\theta_{(A,\alpha)} \neq \pm 1$ .

### 3.1.3 Anti-particles

In DGTs every particle-type  $(A, \alpha)$  has a unique conjugate particle-type  $(\bar{A}, \bar{\alpha})$  with the special property that, as a pair, they can fuse to the vacuum, which is formalized in the fusion rule

$$\Pi_\alpha^A \otimes \Pi_{\bar{\alpha}}^{\bar{A}} = \Pi_0^{\{e\}} + \dots \quad (3.27)$$

As a representation of  $D[H]$ , we can give the structure of  $(\bar{A}, \bar{\alpha})$  explicitly. The representation module of  $(\bar{A}, \bar{\alpha})$  is just the dual of the representation module of  $(A, \alpha)$ , i.e.  $V_{\bar{\alpha}}^{\bar{A}} = (V_\alpha^A)^*$ . The action of  $P_h g$  on a state  $\langle ^A h_i, \alpha v_j |$  is given by

$$\Pi_{\bar{\alpha}}^{\bar{A}}(P_h g): \langle ^A h_i, \alpha v_j | \rightarrow \langle ^A h_i, \alpha v_j | \Pi_\alpha^A(P_{g^{-1}h^{-1}g} g^{-1}) \quad (3.28)$$

Indeed, we do find a one-dimensional subspace in  $V_\alpha^A \otimes V_{\bar{\alpha}}^{\bar{A}}$  that transforms like the vacuum under the action of  $D[H]$ , namely the subspace spanned by

$$\sum_{i,j} |^A h_i, \alpha v_j\rangle \langle ^A h_i, \alpha v_j| \quad (3.29)$$

We will show that the subspace spanned by this element transforms trivially, below.

Recall that the vacuum representation is given by

$$\Pi_0^{\{e\}}(P_h g) = \epsilon(P_h g) = \delta_{h,e} \quad (3.30)$$

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Now note that, in general, elements of  $V_\alpha^A \otimes V_{\bar{\alpha}}^A$  can be regarded as linear operators on  $V_\alpha^A$ . In particular we have

$$\sum_{i,j} |{}^A h_i, \alpha_{v_j}\rangle \langle {}^A h_i, \alpha_{v_j}| = \mathbf{1}_{(A,\alpha)} \quad (3.31)$$

where  $\mathbf{1}_{(A,\alpha)}$  denotes the identity operator on  $V_\alpha^A$ . In particular it commutes with the representation matrices. From this, it is easy to deduce that, as a state, it transforms like the vacuum as claimed. Indeed, working out the action of  $P_h g$  on  $\mathbf{1}_{(A,\alpha)}$  gives

$$\begin{aligned} \sum_{h'h''=h} \Pi_\alpha^A(P_{h'} g) \mathbf{1}_{(A,\alpha)} \Pi_\alpha^A(P_{g^{-1}h''^{-1}g} g^{-1}) &= \sum_{h'h''=h} \delta_{h',h''^{-1}} \Pi_\alpha^A(P_{h'} g^{-1}g) \mathbf{1}_{(A,\alpha)} \\ &= \sum_{h'} \delta_{h,e} \Pi_\alpha^A(P_{h'} e) \mathbf{1}_{(A,\alpha)} \end{aligned} \quad (3.32)$$

Note the subtle difference between  $(P_h g)^*$  and  $S(P_h g)$ .

Again, it is a general feature that quantum group symmetry incorporates anti-particles in a natural way. For this we need a linear map  $S$  from the quantum group to itself, which is called the antipode, satisfying

$$\sum_{(a)} S(a') a'' = 1 \epsilon(a) = \sum_{(a)} a' S(a'') \quad (3.33)$$

for all elements  $a$  of the quantum group. Here we have made use of the Sweedler notation

$$\Delta(a) = \sum_{(a)} a' \otimes a'' \quad (3.34)$$

which generalizes

$$\Delta(P_h g) = \sum_{h'h''} P_{h'} g \otimes P_{h''} g \quad (3.35)$$

to arbitrary cases.

### 3.2 General remarks

The discussion above is included to give a general idea of quantum group symmetry. It is possible to give rigorous and general definitions of the essential structures that appeared above, and to define in general what we mean by a quantum group. Let us give the usual mathematical nomenclature and some relevant references. Suppose we start with an associative algebra  $\mathcal{H}$  that has a unit  $1 \in \mathcal{H}$ . Definition of a comultiplication  $\Delta$  and counit  $\epsilon$  satisfying certain axioms gives a coalgebra structure, which, if it is compatible with the algebra structure, makes  $\mathcal{H}$  a bialgebra. If there is an antipode, this makes it a Hopf algebra. It can be shown that the antipode, if it exists, is unique, such that any bialgebra has at most one Hopf algebra structure. The involution or  $*$ -structure necessary to define unitary representations makes it a Hopf- $*$ -algebra.

The most important feature in relation to planar physics is the possibility of non-trivial braiding. This is given by the existence of a universal  $R$ -matrix, which is an element of  $\mathcal{H} \otimes \mathcal{H}$ . It has to satisfy certain consistency relations which ensure that the braiding of representations obey the Yang-Baxter equation. The name for a Hopf algebra with universal  $R$ -matrix is quasi-triangular.

It is nice to contemplate what is really new for quantum groups in comparison with the usual group symmetries abundant in physics. Let us consider two important cases, symmetries given by a finite group  $H$  and symmetries given by a continuous Lie group  $G$ .

instead of the group  $H$ , we can alternatively work with the group algebra  $\mathbb{C}[H]$  without losing any information. This is just the complex algebra generated by the group elements with the multiplication given by the group operation.

$$\mathbb{C}[H] = \bigoplus_{g \in H} \mathbb{C}g, \quad \left( \sum_g c_g g \right) \cdot \left( \sum_{g'} c_{g'} g' \right) = \sum_{g.g'} c_g c_{g'} gg' \quad (3.36)$$

This is in fact a Hopf algebra with comultiplication, counit and antipode

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1} \quad (3.37)$$

It is quasi-triangular, with universal  $R$ -matrix

$$\mathcal{R} = e \otimes e \quad (3.38)$$

where  $e$  is the unit element of  $H$ . Since this universal  $R$ -matrix is trivial, braiding just amount to flipping the tensor legs. This gives rise to the usual particle statistics in higher dimensions. We might call  $\mathbb{C}[H]$  triangular, instead of quasi-triangular.

For Lie groups  $G$  the situation is similar. Instead of the Lie group, one usually works with the Lie algebra  $\mathfrak{g}$ . We usually allow for normal products  $xy$  apart from the bracket  $[x, y]$  and incorporate a unit 1 element, which is very natural if we think of this Lie algebra as an algebra of symmetry operators. Formally, this leads to the universal enveloping algebra  $U[\mathfrak{g}]$ , which is generated by the unit 1 and the elements of  $\mathfrak{g}$  subject to the relations

$$xy - yx = [x, y] \quad (3.39)$$

It becomes a (quasi-)triangular Hopf algebra by defining

$$\begin{aligned} \Delta(1) &= 1 \otimes 1 & \Delta(x) &= x \otimes 1 + 1 \otimes x \\ \epsilon(1) &= 1 & \epsilon(x) &= 0 \\ S(1) &= 1 & S(x) &= -x \\ \mathcal{R} &= 1 \otimes 1 \end{aligned} \quad (3.40)$$

which can be seen as the infinitesimal version of the definitions given for  $\mathbb{C}[H]$ .

When we use the term quantum group, we have quasi-triangular Hopf algebras in mind, with a *non-trivial*  $R$ -matrix leading to interesting braiding. Some authors

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seem to prefer to use the term quantum group only for quasi-triangular Hopf algebras that occur as  $q$ -deformations of semi-simple Lie algebras discussed below, or to include quasi-Hopf algebras, for which the coproduct does not precisely satisfy the axioms for a coalgebra, but do almost. For more general information on quantum groups and precise definitions, we refer to the literature, for example [46, 60].

#### 3.3 Chern-Simons theory and $U_q[\mathfrak{su}(2)]$

The quantum doubles that can be obtained via the Drinfel'd construction form an important class of examples of quantum groups. The other important class comes from a semi-simple Lie algebra by “ $q$ -deformation” of the universal enveloping algebra. We will discuss an important example,  $U_q[\mathfrak{su}(2)]$ , that comes from the Lie algebra  $\mathfrak{su}(2)$ . It plays an important role in the quantization of Chern-Simons theory with gauge group  $SU(2)$  [84] and in the Wess-Zumino-Witten models for CFT [79, 82] based on the affine algebra of  $\mathfrak{su}(2)$  at level  $k$ . We follow [72].

One can view  $U_q[\mathfrak{su}(2)]$  as the algebra generated by the unit 1 and the three elements  $H, L^+$  and  $L^-$ , which satisfy the relations

$$[H, L^\pm] = \pm 2L^\pm \quad (3.41)$$

$$[L^+, L^-] = \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{-1/2}} \quad (3.42)$$

where  $q$  is a formal parameter that may be set to any non-zero complex number. The coproduct  $\Delta$  is given on the generators by

$$\Delta(1) = 1 \otimes 1 \quad (3.43)$$

$$\Delta(H) = 1 \otimes H + H \otimes 1 \quad (3.44)$$

$$\Delta(L^\pm) = L^\pm \otimes q^{H/4} + q^{-H/4} \otimes L^\pm \quad (3.45)$$

The counit and antipode are

$$\epsilon(1) = 1, \quad \epsilon(H) = 0, \quad \epsilon(L^\pm) = 0 \quad (3.46)$$

$$S(1) = 1, \quad S(H) = -H, \quad S(L^\pm) = -q^{\mp 1/4} L^\pm \quad (3.47)$$

There is a  $*$ -structure given by

$$(L^\pm)^* = L^\mp, \quad H^* = H \quad (3.48)$$

giving rise to the notion of unitary representations. Without the  $*$ -structure, it is perhaps better to call this algebra  $U_q[\mathfrak{sl}(2)]$  instead of  $U_q[\mathfrak{su}(2)]$ , under which name this quantum group usually appears in the literature.

In the limit  $q \rightarrow 1$ , we recover the definitions for the universal enveloping  $U[\mathfrak{su}(2)]$  algebra of  $\mathfrak{su}(2)$ , for instance

$$[L^+, L^-] = H \quad (3.49)$$

Therefore, it is sensible to talk about a  $q$ -deformation of  $U[\mathfrak{su}(2)]$ .

If  $q$  is not a root of unity, the representation theory is very similar to that of  $U[\mathfrak{su}(2)]$ . For every half-integer  $j \in \frac{1}{2}\mathbb{Z}$  there is an irreducible highest weight representation of dimension  $d = 2j + 1$  with highest weight  $\lambda = 2j$ . The representation modules  $V^\lambda$  have an orthonormal basis

$$|j, m\rangle, \quad m \in \{-j, -j + 1, \dots, j - 1, j\} \quad (3.50)$$

The action of the generators is

$$\Pi^\lambda(H) |j, m\rangle = 2m |j, m\rangle \quad (3.51)$$

$$L^\pm |j, m\rangle = \sqrt{[j \mp m]_q [j \pm m \pm 1]_q} |j, m \pm 1\rangle \quad (3.52)$$

Here we have used the notation

$$[m]_q = \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}} \quad (3.53)$$

for the so called  $q$ -numbers that enter the formula to the commutation relation  $[L^+, L^-] = [H]_q$ .

A formal expression for the universal  $R$ -matrix of  $U_q[\mathfrak{su}(2)]$  is

$$\mathcal{R} = q^{\frac{H \otimes H}{4}} \sum_{n=0}^{\infty} \frac{(1 - q^{-1})^n}{[n]_q!} q^{n(1-n)/4} \left( q^{nH/4} (L^+)^n \right) \otimes \left( q^{-nH/4} (L^-)^n \right) \quad (3.54)$$

From this expression, one can work out the action on representations. This produces a non-trivial braiding in a similar fashion as we saw for DGT and the quantum double.

In relation to Chern-Simons theory and the WZW models, the most interesting case is, however, when  $q$  is not a root of unity. In that case, the representation theory of  $U_q[\mathfrak{su}(2)]$  changes drastically. We will not go in to the details of the mathematical tricks needed to find the right notion representations that describe charges in these physical theories. These can be read, for instance, in [46, 60] or [72], but the right definition was first discovered in [42]. In this case, we denote the resulting theory as  $\mathfrak{su}(2)_k$ , where  $k$  is related to  $q$  as  $q = \exp(\frac{2\pi i}{k+2})$ . We will come back to these theories in chapter 5.

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## CHAPTER 4

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### Anyons and tensor categories

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As discussed in the previous chapter, quantum group symmetry can account for fusion and non-trivial braiding of excitations in (2+1)-dimensional systems. But the physical states usually correspond to the unitary irreps rather than the internal states of the representation modules. Also, in the case of  $U_q[\mathfrak{su}(2)]$  modules for  $q$  a root of unity, the identification of representations with particle types is not straight forward. One may ask if a rigorous formalism exists that treats these theories on the level of the excitations rather than the underlying symmetries. The answer is yes.

Mathematically, the representations of a quantum group (often) form a modular tensor category, a structure that may be defined independently. In this chapter we introduce many generalities of these tensor categories as they arise in physical applications. These capture the topological properties, including the addition of quantum numbers (fusion), of many theories in a very general way and allow for a graphical formalism which can be regarded as a kind of topological Feynman diagrams. They can be constructed from quantum groups, but also arise directly from a (rational) CFT and can be used describe the excitations in Kitaev and Levin-Wen type lattice models directly. They are generally related to topological quantum field theories.

In stead of relying heavily on the language of category theory, we have chosen a route along the lines of [22, 66]. This means that we focus on concreteness and calculability, in favour of formal language. For the mathematically inclined, this basically means that we assume the category to be strict and construct everything in terms of simple objects. A paper in the mathematical physics literature that treats tensor categories in a similar fashion is [34] where symmetries of the  $F$ -symbols (to be defined below) are derived. Proper mathematical textbooks are [14, 75].

We have particularly relied on [22] for this chapter. In appendix B we give a more mathematically formulated description of the kind of categories that we consider, and point out how this connects to the formulation below.

### 4.1 Fusing and splitting

For a general theory, the particle-like excitations fall in different topological classes or sectors. For simplicity, we treat these sectors as elementary excitations and assume there is finite number of them. These are the first ingredients of a modular tensor category, which in a physical context has to be unitary, or, more generally, a braided tensor category.

One can vary the assumptions on the kind of tensor category in consideration. Each set of assumptions gives a slightly different mathematical structure which has its own name. To avoid getting stuck in nomenclature, we will generally refer to a ‘the theory’ in this chapter. The term *anyon models* has also appeared in the physical literature for unitary braided tensor categories.

#### 4.1.1 Fusion multiplicities

We start with a theory  $\mathcal{A}$  with a finite collection of topological sectors  $C_{\mathcal{A}} = \{a, b, c, \dots\}$ . We also refer to these sectors as particle-types, (anyonic) charges, (particle) labels or sometimes simply particles or anyons. They obey a set of fusion rules that we write as

$$a \times b = \sum_{c \in C_{\mathcal{A}}} N_{ab}^c c \quad (4.1)$$

(The domain of the sum will be left implicit from now on.) Here the  $N_{ab}^c$  are non-negative integers called fusion multiplicities. These determine the possibilities for the total charge when two anyons are combined (fused). The total charge of two anyons with respective labels  $a$  and  $b$  can be any of the  $c$  with  $N_{ab}^c \neq 0$ . If  $N_{ab}^c \neq 0$ , we also speak of the fusion channel  $c$  of  $a$  and  $b$ , and we will sometimes write this as  $c \in a \times b$ .

If the state-space of a pair of particles is multidimensional this gives rise to non-Abelian anyons.<sup>1</sup> We say, therefore, that the theory is non-Abelian if there are charges  $a$  and  $b$  that have

$$\sum_c N_{ab}^c > 1 \quad (4.2)$$

The particle labels together with the fusion rules (4.1) generate the *fusion algebra* of the theory. In this context we sometimes denote the basis states by kets  $|a\rangle, |b\rangle, |c\rangle$ .

The fusion algebra has to obey certain conditions which have a clear physical interpretation. We require that there is a unit, i.e. a unique trivial particle, the vacuum,  $0 \in \mathcal{A}$  that fuses trivially with all labels:  $0 \times a = a = a \times 0$  for all  $a$ . Furthermore, fusion should be an associative operation,  $(a \times b) \times c = a \times (b \times c)$ , such that the total anyonic charge is a well-defined notion. In terms of the fusion multiplicities, these conditions

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<sup>1</sup>There is a subtlety to the term non-Abelian anyons. Even if a pair of identical particles have a multidimensional state space, their braiding might still be Abelian. Hence, one could say that this is a necessary but not sufficient condition, but we will not be bothered with the distinction.

can translate to

$$N_{0a}^b = N_{a0}^b = \delta_{ab} \quad (4.3)$$

$$\sum_e N_{ab}^e N_{ec}^d = \sum_f N_{af}^d N_{bc}^f \quad (4.4)$$

We also require that the fusion rules are commutative, such that  $a \times b = b \times a$ . This should hold for (2+1)-dimensional systems, since there is no way to define left and right unambiguously. We can lift this requirement if we consider charges of a (1+1)d system, for example when we look at boundary excitations. Finally, each charge  $a \in C_{\mathcal{A}}$  should have a unique conjugate charge  $\bar{a} \in C_{\mathcal{A}}$  or anti-particle such that  $a$  and  $\bar{a}$  can annihilate

$$a \times \bar{a} = 0 + \sum_{c \neq 0} N_{ab}^c c \quad (4.5)$$

This does not mean that  $a$  and  $\bar{a}$  can only fuse to the vacuum, as there might be  $c \neq 0$  with  $N_{a\bar{a}}^c \neq 0$ . Note that  $\bar{\bar{a}} = a$ .

The following relations for the fusion multiplicities also follow from these conditions on fusion

$$N_{ab}^0 = \delta_{b\bar{a}} \quad (4.6)$$

$$N_{ab}^c = N_{ba}^c = N_{b\bar{c}}^{\bar{a}} = N_{\bar{a}\bar{c}}^{\bar{c}} \quad (4.7)$$

The first line is just the anti-particle property. The second line can be derived using commutativity and associativity.

It is useful to define the fusion matrices  $N_a$  with

$$(N_a)_{bc} = N_{ab}^c \quad (4.8)$$

The fusion rules give

$$N_a N_b = \sum_c N_{ab}^c N_c \quad (4.9)$$

which shows that the fusion matrices provide a representation of the fusion algebra. This is just the representation of the fusion algebra that is obtained by letting it act on itself via the fusion product. Associativity of the fusion rules translates to commutativity of the fusion matrices, such that

$$N_a N_b = N_b N_a \quad (4.10)$$

The explicit diagonalization of the fusion matrices is a very interesting result, that we come back to in section 4.4.

### 4.1.2 Diagrams and $F$ -symbols

Special about the formalism based on category theory in comparison with the quantum group approach is that internal states are totally left out of the picture. This makes

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sense because, in physical settings, to obtain the gauge invariant states, we still have to mod out by the quantum group. Operators on anyons are best described in a diagrammatic formalism.<sup>1</sup> They form states in certain vector spaces. The formalism gives rules to manipulate the diagrams, and perform diagrammatic calculations. Manipulating diagrams according is a way to rewrite the same state, for example, by changing to another basis. One alters the representation, but not the state (operator) itself.

Each anyonic charge label is associated with a directed line. This is the identity operator for the anyon, or in the language of category theory, the identity morphism. It is often useful to think of it as the world line of the anyon propagating in time, which we will take as flowing upward. Reversing the orientation of a line segment is equivalent to conjugating the charge labelling it, so that

$$\begin{array}{c} \uparrow \\ a \end{array} = \begin{array}{c} \downarrow \\ \bar{a} \end{array} \quad (4.11)$$

The second most elementary operators are directly related to fusion and the dual process, splitting of anyons. To every triple of charge labels  $(a, b, c)$ , assign a complex vector space  $V_{ab}^c$  of dimension  $N_{ab}^c$ . This is called a *fusion space*. States of the fusion space  $V_{ab}^c$  are denoted by *fusion vertices* with labels corresponding to the charges on the outer legs

$$\begin{array}{c} \uparrow c \\ \mu \\ \swarrow a \quad \searrow b \end{array}, \quad \mu = 1, \dots, N_{ab}^c \quad (4.12)$$

The dual of the fusion space is denoted  $V_c^{ab}$  and is called a *splitting space*. The states of the splitting space  $V_c^{ab}$  are denoted by *splitting vertices* like

$$\begin{array}{c} \swarrow a \quad \searrow b \\ \mu \\ \uparrow c \end{array}, \quad \mu = 1, \dots, N_{ab}^c \quad (4.13)$$

When  $N_{ab}^c = 1$  we can leave the index  $\mu$  implicit.<sup>2</sup>

The 0-line can be inserted and removed from diagrams at will, reflecting the properties of the vacuum, and is therefore often left out or ‘invisible’. When made explicit, we draw vacuum lines dotted. Hence, we have

$$\begin{array}{c} \text{---} 0 \\ \swarrow a \\ \uparrow a \end{array} = \begin{array}{c} \uparrow \\ a \end{array} = \begin{array}{c} \swarrow a \\ \uparrow a \\ \text{---} 0 \end{array} \quad (4.14)$$

and similar for the corresponding fusion vertices.

In terms of quantum group representations, splitting vertices correspond to the embedding of an irrep into a tensor product representation, while fusion vertices should

<sup>1</sup>In the mathematical literature this is known as graphical calculus, which was introduced by Reshetikhin and Turaev [70].

<sup>2</sup>In subsequent chapters, we will assume that all  $N_{ab}^c = 0, 1$ . Therefore, diagrams will have no labels at the vertices, but only at the charge lines. For completeness, however, we have left the  $\mu$ 's in tact in this chapter.

be thought of as the projection onto an irreducible subspace of a tensor product representation.

General anyon operators can be made by stacking fusion and splitting vertices and taking linear combinations. The intermediate charges of connected charge lines have to agree and the vertices should be allowed by the fusion rules, otherwise the whole diagram evaluates to zero. This gives a diagrammatic encoding of charge conservation. As a general notation we write  $V_{a'_1, \dots, a'_n}^{a_1, \dots, a_m}$  for the vector space of operators that take  $n$  anyons with charges  $a'_1, \dots, a'_n$  as input and which produce  $m$  anyons of charges  $a_1, \dots, a_m$ .

An important example is  $V_d^{abc}$ . Operators of this space can be made by stacking two splitting vertices on top of each other. The choice we have of connecting the top vertex either on the right or on the left of the bottom vertex leads to two different bases of  $V_d^{abc}$ . The change of basis in these spaces is given by so called  $F$ -symbols<sup>1</sup>  $[F_d^{abc}]_{(e, \alpha, \beta)(f, \mu, \nu)}$ , which are an important piece of data for these models. They are defined by the diagrammatic equation

$$\begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagup \\ \alpha \quad e \quad \beta \\ \diagup \quad \diagdown \quad \diagup \\ d \end{array} = \sum_{f, \mu, \nu} [F_d^{abc}]_{(e, \alpha, \beta)(f, \mu, \nu)} \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagup \\ \nu \quad f \quad \mu \\ \diagup \quad \diagdown \quad \diagup \\ d \end{array} \quad (4.15)$$

Like all other diagrammatic equations in the formalism, this ‘ $F$ -move’ can be used locally, within bigger diagrams to perform calculations. If a diagram on the right is not permitted by the fusion rules, we put the corresponding  $F$ -symbol to zero.

The  $F$ -symbols have to satisfy certain consistency conditions, called the pentagon relations, that we will come back to later. There is a certain gauge freedom in the  $F$ -symbols, reflecting the fact that we can perform a unitary change of basis in the elementary splitting space  $V_c^{ab}$  without changing the theory, which will also be discussed later.

We could just as well have introduced the  $F$ -symbols in terms of fusion states in  $V_{abc}^d$ . The  $F$ -move for fusion states is governed by the adjoint of the  $F$ -move as introduced above. Unitarity of the model amounts to

$$[(F_d^{abc})^\dagger]_{(f, \mu, \nu)(e, \alpha, \beta)} = [F_d^{abc}]_{(e, \alpha, \beta)(f, \mu, \nu)}^* = [(F_d^{abc})^{-1}]_{(f, \mu, \nu)(e, \alpha, \beta)} \quad (4.16)$$

This can be seen by taking the adjoint of the diagrammatic equation for the  $F$ -symbols. In general, the adjoint of a diagram in a unitary theory is taken by reflecting it in the horizontal plane, such that top and bottom interchange, and then reversing the orientation of all arrows.

If one of the upper outer legs of the tree occurring in the  $F$ -move equation is labelled by the vacuum, it is essentially just a splitting vertex and the  $F$ -move should leave it unchanged. Thus, for example  $[F_d^{0bc}]_{ef} = \delta_{eb} \delta_{fc}$ , and similarly if the middle or right upper index equals 0. Note that when  $d = 0$  there is only one non zero  $F$ -symbol for fixed  $a, b, c$ , namely  $[F_0^{abc}]_{\bar{c}\bar{a}}$ , but this can be non-trivial.

<sup>1</sup>The  $F$ -symbols are sometimes called recoupling coefficients or quantum  $6j$ -symbols. The  $F$ -symbols are the analogue of the  $6j$ -symbols from the theory of angular momentum, hence the latter name.

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The pairing of  $V_c^{ab}$  and  $V_{ab}^c$  or the inner product of fusion/splitting states, is again denoted by stacking the appropriate diagrams. We think of the elements of  $V_c^{ab}$  as kets and of the elements of  $V_{ab}^c$  as bras. Composition of operators is written from bottom to top, in accordance with the flow of time. The conservation of anyonic charge is also taken into account in the diagrams, which gives

$$\begin{array}{c} c \\ \uparrow \\ a \quad \mu \\ \swarrow \quad \searrow \\ \mu' \\ \downarrow \\ c' \end{array} b = \delta_{cc'} \delta_{\mu\mu'} \sqrt{\frac{d_a d_b}{d_c}} \left| \begin{array}{c} c \\ \uparrow \\ a \\ \downarrow \\ c \end{array} \right. \quad (4.17)$$

Here, the *quantum dimension*  $d_a$  is defined for each particle label  $a$  as

$$d_a = |[F_a^{a\bar{a}a}]_{00}|^{-1} \quad (4.18)$$

This is a very important quantity in all that follows. From the properties of the  $F$ -symbol immediately find that  $d_0 = 1$ . The normalization factor  $\sqrt{\frac{d_a d_b}{d_c}}$  is inserted in this ‘diagrammatic bracket’ to make the diagrams invariant under isotopy, i.e. under bending of the charge lines (end points should be left fixed).

For horizontal bending, keeping in tact the flow of charge in time, isotopy invariance is trivial. Vertical bending however, introduces events of particle creation and annihilation. Invariance under this kind of bending is almost realized by introducing the normalization of the vertex bracket, but a subtle issue remains.

Consider the calculation

$$\begin{array}{c} \uparrow \\ \circ \\ \swarrow \quad \searrow \\ \bar{a} \\ \downarrow \\ \circ \\ \uparrow \end{array} a = [F_a^{a\bar{a}a}]_{00} \begin{array}{c} \uparrow \\ \circ \\ \swarrow \quad \searrow \\ \bar{a} \\ \downarrow \\ \circ \\ \uparrow \end{array} a = d_a [F_a^{a\bar{a}a}]_{00} \left| \begin{array}{c} a \\ \uparrow \\ a \\ \downarrow \\ a \end{array} \right. \quad (4.19)$$

which uses the  $F$ -move and (4.17). By the definition of  $d_a$ , the coefficient on the right has unit norm. However,  $[F_a^{a\bar{a}a}]_{00}$  might have a non-trivial phase

$$\mathbf{fb}_a = d_a [F_a^{a\bar{a}a}]_{00} \quad (4.20)$$

As we will see later, for  $a \neq \bar{a}$  the phase  $\mathbf{fb}_a = \mathbf{fb}_{\bar{a}}^*$  can be set to unity by a gauge transformation. But when  $a$  is self dual,  $\mathbf{fb}_a = \pm 1$  is a gauge invariant quantity known as the Frobenius-Schur indicator. Hence, in the appropriate gauge, isotopy invariance is realized up to a sign.

To account for this sign, we introduce cup and cap diagrams with a direction given by a flag. These are defined as

$$\begin{array}{c} \uparrow \\ \curvearrowright \\ \bar{a} \end{array} = \mathbf{fb}_a \begin{array}{c} \uparrow \\ \curvearrowright \\ \bar{a} \end{array} = \begin{array}{c} \uparrow \\ \searrow \quad \swarrow \\ \circ \\ \downarrow \end{array} \quad (4.21)$$

and

$$\begin{array}{c} \uparrow \\ \curvearrowleft \\ \bar{a} \end{array} = \mathbf{fb}_a \begin{array}{c} \uparrow \\ \curvearrowleft \\ \bar{a} \end{array} = \begin{array}{c} \uparrow \\ \swarrow \quad \searrow \\ \circ \\ \downarrow \end{array} \quad (4.22)$$

Bending a line vertically is now taken to include the introduction of a cap/cup pair with oppositely directed flags. Then we have the following equality

$$\begin{array}{c} \curvearrowright \\ \uparrow a \end{array} = \begin{array}{c} \uparrow \\ a \end{array} = \begin{array}{c} \downarrow \\ \curvearrowleft \\ a \end{array} \tag{4.23}$$

With these conventions in place, isotopy invariance of the diagrams is realized. If the cups and caps are paired up with opposite flags in diagrams, we leave them implicit. This, shows that loops evaluate as the corresponding quantum dimension

$$\begin{array}{c} a \\ \circlearrowleft \end{array} = d_a \tag{4.24}$$

when we combine this with relation (4.17).

In accordance with the normalization for the inner product, the identity operator on a pair of anyons can now be written as

$$\begin{array}{c} \uparrow \\ a \end{array} \begin{array}{c} \uparrow \\ b \end{array} = \sum_{c,\mu} \sqrt{\frac{d_c}{d_a d_b}} \begin{array}{c} a \quad b \\ \quad \mu \\ c \quad \mu \\ a \quad b \end{array} \tag{4.25}$$

It is very convenient to define  $F$ -symbols for diagrams with two-anyon input and two-anyon output. This is done by the diagrammatic equations

$$\begin{array}{c} a \quad b \\ \alpha \quad e \quad \beta \\ c \quad d \end{array} = \sum_{(f,\mu,\nu)} [F_{cd}^{ab}]_{(e,\alpha,\beta)(f,\mu,\nu)} \begin{array}{c} a \quad b \\ \quad \mu \\ f \quad \nu \\ c \quad d \end{array} \tag{4.26}$$

Equation (4.25) gives

$$[F_{ab}^{ab}]_{0,(c,\mu,\nu)} = \sqrt{\frac{d_c}{d_a d_b}} \delta_{\mu,\nu} \tag{4.27}$$

And more generally we have

$$[F_{cd}^{ab}]_{(e,\alpha,\beta)(f,\mu,\nu)} = \sqrt{\frac{d_e d_f}{d_a d_d}} [F_f^{ceb}]_{(a,\alpha,\mu)(d,\beta,\nu)}^* \tag{4.28}$$

These alternative  $F$ -moves can be used to change a splitting vertex with one leg bend down into a fusion vertex, etcetera. This gives equalities like

$$\begin{array}{c} 0 \\ \bar{a} \quad c \\ \mu \\ a \quad b \end{array} = \sum_{\mu'} [F_{ab}^{0c}]_{(\bar{a},\mu)(c,\mu')} \begin{array}{c} c \\ \mu' \\ a \quad b \end{array} \tag{4.29}$$

and

$$\begin{array}{c} a \quad b \\ \mu \quad \bar{b} \\ c \quad 0 \end{array} = \sum_{\mu'} [F_{c0}^{ab}]_{(\bar{a},\mu)(c,\mu')} \begin{array}{c} a \quad b \\ \mu' \\ c \end{array} \tag{4.30}$$

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By looking at the adjoint versions, more equalities may be derived.

For the general spaces of operators  $V_{a'_1, \dots, a'_n}^{a_1, \dots, a_m}$  one usually agrees on a standard basis. The choice generally made is the basis made by stacking vertices on the left, i.e. consisting of operators

$$(4.31)$$

Any operator in the theory can in principle be expressed in terms of the standard basis elements. This is a very important remark. It implies, for example, that any operator we write down that acts on an anyon of charge  $a$  is, effectively, nothing but multiplication by a complex number since the space  $V_a^a$  is one-dimensional. Furthermore, many equalities can be deduced by comparing coefficients of the corresponding equations expressed in terms of the standard basis. But for other applications, it is sometimes tedious to write everything in terms of the standard basis, and it is much more convenient to leave the diagrams intact.

### 4.1.3 Gauge freedom

As remarked before, there is a certain freedom present in any anyon model. This corresponds to the choice of bases in the  $V_c^{ab}$ . We can apply a unitary transformations in the spaces  $V_c^{ab}$  without changing the theory. Denote such a unitary transformations by  $u_c^{ab}$ . This gives new basis states

$$(4.32)$$

The effect of this transformation on the  $F$ -symbols is

$$[F_d^{abc}]_{(e, \alpha', \beta')(f, \mu', \nu')} = \sum_{\alpha, \beta, \mu, \nu} [(u_e^{ab})^{-1}]_{\alpha' \alpha} [(u_d^{ec})^{-1}]_{\beta' \beta} [F_d^{abc}]_{(e, \alpha, \beta)(f, \mu, \nu)} [u_f^{bc}]_{\mu' \mu} [u_d^{af}]_{\nu' \nu} \quad (4.33)$$

When there are no multiplicities, the transformations  $u_c^{ab}$  are just complex phases. In this case, which is of most practical interest, we have the freedom to redefine the  $F$ -symbols following

$$[F_d^{abc}]'_{ef} = \frac{u_f^{bc} u_d^{af}}{u_e^{ab} u_d^{ec}} [F_d^{abc}]_{ef} \quad (4.34)$$

We can use this freedom to switch to a more convenient set of  $F$ -symbols. Note that we do not allow  $u_a^{0a}$  and  $u^{a0}$  to differ from unity due to (4.14).

From (4.34) it is straight forward that we can remove the phase  $\mathbf{fb}_a$  of  $[F_a^{a\bar{a}a}]_{00}$  for all  $a \neq \bar{a}$ , by taking  $u_0^{a\bar{a}} = \mathbf{fb}_a$  and  $u_0^{\bar{a}a} = 1$ . So there is a choice of gauge for which all Frobenius-Schur indicators are equal to unity, except for self-conjugate particles as claimed before.

The gauge freedom discussed here should not be mistaken for gauge freedom in the sense of gauge theories. We only wish to express that there is a redundancy in the theory.

#### 4.1.4 Tensor product and quantum trace

The tensor product of two operators acting on well separated groups of anyons is made by juxtaposition of the corresponding diagrams. For a general operator we introduce the notation

$$\begin{array}{c}
 A_1 \quad A_m \\
 \uparrow \quad \dots \quad \uparrow \\
 \boxed{X} \\
 \uparrow \quad \dots \quad \uparrow \\
 A'_1 \quad A'_n
 \end{array}
 = X \in V_{A'_1 \dots A'_n}^{A_1 \dots A_m} = \bigoplus_{\substack{a_1 \dots a_m \\ a'_1 \dots a'_n}} V_{a'_1 \dots a'_n}^{a_1 \dots a_m} \quad (4.35)$$

where we used capital labels to denote sums over all anyonic charges such that  $X$  is defined to act on any input of  $n$  anyons, giving some  $m$  anyon output. The tensor product of two operators  $X$  and  $Y$  is then given by the diagram

$$\begin{array}{c}
 \uparrow \quad \dots \quad \uparrow \uparrow \quad \dots \quad \uparrow \\
 \boxed{X \otimes Y} \\
 \uparrow \quad \dots \quad \uparrow \uparrow \quad \dots \quad \uparrow
 \end{array}
 = \begin{array}{c}
 \uparrow \quad \dots \quad \uparrow \\
 \boxed{X} \\
 \uparrow \quad \dots \quad \uparrow
 \end{array}
 \begin{array}{c}
 \uparrow \quad \dots \quad \uparrow \\
 \boxed{Y} \\
 \uparrow \quad \dots \quad \uparrow
 \end{array} \quad (4.36)$$

While it is of course possible to rewrite this in terms of the standard basis, it is usually much more convenient to keep the separation explicit.

In the mathematical literature, one often sees equation (4.24) as the definition of the quantum dimension. It is a special case of the quantum trace applied to the identity operator. The quantum trace of a general operator  $X$ , denoted as  $\widehat{\text{Tr}} X$ , is formed diagrammatically by closing the diagram with loops that match the outgoing lines with the incoming lines at the same position

$$\widehat{\text{Tr}} X = \widehat{\text{Tr}} \begin{array}{c}
 \left[ \begin{array}{c}
 A_1 \quad A_n \\
 \uparrow \quad \dots \quad \uparrow \\
 \boxed{X} \\
 \uparrow \quad \dots \quad \uparrow \\
 A'_1 \quad A'_n
 \end{array} \right] = \begin{array}{c}
 A_1 \quad \dots \quad A_n \\
 \uparrow \quad \dots \quad \uparrow \\
 \boxed{X} \\
 \uparrow \quad \dots \quad \uparrow
 \end{array}
 \end{array} \quad (4.37)$$

The fact that the quantum dimension is the quantum trace of the identity operator is analogous to the fact that the trace of the identity matrix gives the dimension of the vector space it acts on. In cases where we can speak of an internal Hilbert space of the

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anyon, as in DGT, the quantum trace is indeed the same as the trace over this internal space.

From the diagrams it is clear that this definition of trace and tensor product behave in the appropriate way. For two operators  $X$  and  $Y$ , we have  $\widehat{\text{Tr}}(X \otimes Y) = \widehat{\text{Tr}}(X) \widehat{\text{Tr}}(Y)$ .

We can also define the partial trace, and ‘trace out’ a subsystem. In this case only a part of the outer lines is closed with loops, either on the outer left or on the outer right side.

### 4.1.5 Topological Hilbert space

The topological Hilbert space of a collection of anyons with charges  $a_1, \dots, a_n$  with total charge  $c$  is the space  $V_b^{a_1, \dots, a_n}$ . It is interesting to study the asymptotic behaviour of the topological Hilbert space of  $n$  anyons of charge  $a$ . This is effectively given by the quantum dimension  $d_a$ , as discussed below. This discussion is based on [66].

An important relation for the quantum dimension is

$$d_a d_b = a \left( \begin{array}{c} \bigcirc \\ \bigcirc \end{array} \right)_b = \sum_{c, \mu} \sqrt{\frac{d_c}{d_a d_b}} \left( \begin{array}{c} c \\ a \quad b \\ \mu \quad \mu \end{array} \right) = \sum_c N_{ab}^c d_c \quad (4.38)$$

This shows that the  $d_a$  form a one-dimensional representation of the fusion algebra.

Now consider the vector  $|\omega\rangle = \sum_a d_a |a\rangle$  in the fusion algebra. Then relation (4.38) shows that  $|\omega\rangle$  is a common eigenvector of the fusion matrices, and we have

$$N_a |\omega\rangle = d_a |\omega\rangle \quad (4.39)$$

Since the matrices  $N_a$  have non-negative integer entries and  $|\omega\rangle$  has only positive coefficients,  $d_a$  must be the largest eigenvalue. Note that this in particular means that we do not need the  $F$ -symbols to deduce  $d_a$ , but that the fusion rules are sufficient. We can rephrase the above observation as the statement that  $|\omega\rangle$  is a common Perron-Frobenius eigenvector of the  $N_a$ .

The significance for the topological Hilbert space is the following. Suppose we have a collection of  $n$  anyons of charge  $a$  which have total charge  $b$ . Denote the dimension of the topological Hilbert space  $V_{aa\dots a}^b$  by  $N_{aa\dots a}^b$ . We have

$$N_{aa\dots a}^b = \sum_{\{b_i\}} N_{aa}^{b_1} N_{ab_1}^{b_2} \dots N_{ab_{(n-2)}}^b \quad (4.40)$$

The matrix  $N_a$  can be diagonalized as

$$N_a = \frac{d_a}{\mathcal{D}} |\omega\rangle \langle \omega| + \dots \quad (4.41)$$

where

$$\mathcal{D} = \sqrt{\langle \omega | \omega \rangle} = \sqrt{\sum_a d_a^2} \quad (4.42)$$

and we have left out sub-leading terms. So for large  $n$ , we obtain

$$N_{aa\dots a}^b \approx d_a^n d_b / \mathcal{D}^2 \quad (4.43)$$

Therefore we see that  $d_a$  controls the rate of growth for the  $n$  particle topological Hilbert space.

Corresponding to  $|\omega\rangle$ , we have the charge line labelled by  $\omega$ . Since  $d_a = d_{\bar{a}}$ , the labels  $a$  and  $\bar{a}$  appear with identical weights. Therefore we can leave out the orientation of the charge line, and write

$$\left| \omega \right. = \sum_a d_a \left| a \right. \quad (4.44)$$

This gives

$$\omega \bigcirc = \mathcal{D}^2 \quad (4.45)$$

We would like to point out the significance of the label  $\omega$  when we consider a DGT. In the representation theory of groups, one often considers the group algebra  $\mathbb{C}[H]$  of, say, a finite group  $H$  as a representation. We can just let  $H$  act on it via the group operation. It is a well known fact that  $\mathbb{C}[H]$  can be written as a sum of irreducible representation which occur with multiplicity given by the dimension of the irrep,  $\mathbb{C}[h] \simeq \sum_{\alpha} d_{\alpha} V_{\alpha}$ . In fact, the same holds for quantum double representations,  $D[H] \simeq \sum_{(A,\alpha)} d_{(A,\alpha)} V_{\alpha}^A$ . Hence, for these theories  $\omega = D[H]$ .

## 4.2 Braiding

The defining properties of anyonic excitations are their non-trivial spin and braiding properties. These are also incorporated in the formalism by means of diagrammatic relations.

The effect of two anyons switching places in the system is governed by the braiding operators, which are written as

$$R_{ab} = \begin{array}{c} \nearrow \\ \searrow \\ a \quad b \end{array}, \quad R_{ab}^{\dagger} = R_{ab}^{-1} = \begin{array}{c} \nwarrow \\ \swarrow \\ a \quad b \end{array} \quad (4.46)$$

They are defined by a set of  $R$ -symbols which give the effect on basis states of  $V_c^{ab}$ . The diagrammatic relation, or  $R$ -moves, are

$$\begin{array}{c} b \quad a \\ \nearrow \quad \nwarrow \\ \searrow \quad \swarrow \\ \mu \\ c \end{array} = \sum_{\nu} [R_c^{ab}]_{\mu\nu} \begin{array}{c} b \quad a \\ \nwarrow \quad \swarrow \\ \nearrow \quad \nwarrow \\ \nu \\ c \end{array} \quad (4.47)$$

$$(4.48)$$

The full braiding operator can thus be represented as

$$\begin{array}{c} \nearrow \\ \searrow \\ a \quad b \end{array} = \sum_{c,\mu,\nu} \sqrt{\frac{d_c}{d_a d_b}} [R_c^{ab}]_{\mu\nu} \begin{array}{c} b \quad a \\ \nwarrow \quad \swarrow \\ \nearrow \quad \nwarrow \\ \nu \\ c \\ \nearrow \quad \nwarrow \\ \mu \\ a \quad b \end{array} \quad (4.49)$$

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The similar equations hold for the inverse  $R$ -move. Unitarity of the  $R$ -moves amounts to

$$[(R_c^{ab})^{-1}]_{\mu\nu} = [(R_c^{ab})^\dagger]_{\mu\nu} = [R_c^{ab}]_{\nu\mu}^* \quad (4.50)$$

For braiding and fusion to be consistent, the  $R$ -moves have to obey the hexagon equations, discussed below. These relations imply that braiding commutes with fusion, which means that in diagrams lines may be passed over and under vertices. They imply the usual Yang-Baxter relation for braids. Together with triviality of fusion with the vacuum they also imply

$$R_a^{a0} = R_b^{0b} = 1 \quad (4.51)$$

so that the vacuum braids trivial, as it should.

The topological spin  $\theta_a$ , also called twist factor, is associated with a  $2\pi$  rotation of an anyon of charge  $a$  and defined by

$$\theta_a = \theta_{\bar{a}} = d_a^{-1} \widehat{\text{Tr}} R_{aa} = \mathbf{fb}_a [R_1^{\bar{a}a}]^* \quad (4.52)$$

Diagrammatically, it can be used to remove twists from a diagram:

$$\begin{array}{c} \uparrow \\ | \\ \text{twist} \\ | \\ a \end{array} = \theta_a \begin{array}{c} \uparrow \\ | \\ a \end{array} = \begin{array}{c} \text{twist} \\ | \\ \uparrow \\ a \end{array} \quad (4.53)$$

and

$$\begin{array}{c} \uparrow \\ | \\ \text{twist} \\ | \\ a \end{array} = \theta_a^* \begin{array}{c} \uparrow \\ | \\ a \end{array} = \begin{array}{c} \text{twist} \\ | \\ \uparrow \\ a \end{array} \quad (4.54)$$

Because the  $\theta_a$  are not necessarily trivial, it is better to think of the lines in diagrams as ribbons for which a twist is really a non-trivial operation, as illustrated below.



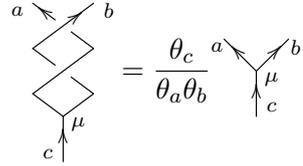
When applicable, the twist factor is related to the (ordinary angular momentum) spin or CFT conformal scaling dimension  $h_a$  of  $a$ , as

$$\theta_a = e^{i2\pi h_a} \quad (4.55)$$

The effect of a double braiding – or monodromy – of two anyons is governed by the *monodromy equation*

$$\sum_{\lambda} [R_c^{ab}]_{\mu\lambda} [R_c^{ba}]_{\lambda\nu} = \frac{\theta_c}{\theta_a \theta_b} \delta_{\mu,\nu} \quad (4.56)$$

which diagrammatically gives

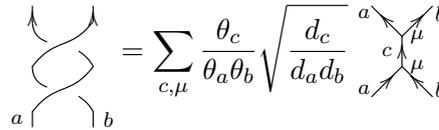


$$= \frac{\theta_c}{\theta_a \theta_b} \text{ (diamond with } a, b, c, \mu \text{)} \quad (4.57)$$

It is a matter of topological manipulation to see that this equation holds. This is more readily seen when we draw the ribbon version



We can also write down the full monodromy operator



$$= \sum_{c, \mu} \frac{\theta_c}{\theta_a \theta_b} \sqrt{\frac{d_c}{d_a d_b}} \text{ (diamond with } a, b, c, \mu \text{)} \quad (4.58)$$

which corresponds to one particle encircling the other, or equivalently a  $2\pi$  rotation of a pair of particles. The monodromy equation, in any form, is used frequently in diagrammatic calculations.

### 4.3 Pentagon and Hexagon relations

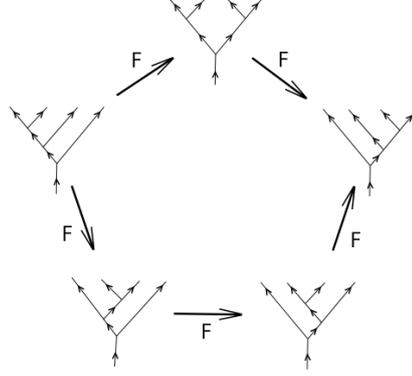
With the definition of fusion rules,  $F$ -symbols, and  $R$ -symbols, the topological data of a theory is completely specified and all diagrams can be calculated. In order to ensure that different ways of manipulating a diagram always lead to the same answer, some consistency conditions on the  $F$ -symbols and  $R$ -symbols have to be satisfied. These are the pentagon and hexagon relations or equations. The MacLane's coherence theorem from category theory guarantees that no other consistency checks are necessary [54].

The pentagon equations are a condition on the  $F$ -symbols. They can be understood as the equivalence of two sequences of moves, as shown in figure 4.1, which can be neatly organized in a pentagon. The resulting equation is

$$\sum_{\delta} [F_e^{fcd}]_{(g, \beta, \gamma)(l, \lambda, \delta)} [F_e^{abl}]_{(f, \alpha, \delta)(k, \mu, \nu)} = \sum_{h, \rho, \sigma, \psi} [F_g^{abc}]_{(f, \alpha, \beta)(h, \rho, \sigma)} [F_e^{ahd}]_{(g, \sigma, \gamma)(k, \psi, \nu)} [F_k^{bcd}]_{(h, \rho, \psi)(l, \lambda, \mu)} \quad (4.59)$$

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**Figure 4.1: The pentagon relations** - The pentagon relation say that the two possible sequences of manipulating the basis states of  $V_e^{abcd}$  using  $F$ -moves give the same result.

The hexagon relations are similarly understood as the equality of two sequences of manipulation. They guarantee consistence of fusion and braiding. The diagrammatic relations are illustrated in figure 4.2. The resulting equations are

$$\sum_{\lambda, \gamma} [R_e^{ca}]_{\alpha\lambda} [F_d^{acb}]_{(e, \lambda, \beta)(g, \gamma, \nu)} [R_g^{cb}]_{\gamma\mu} \quad (4.60)$$

$$= \sum_{f, \sigma, \delta, \psi} [F_d^{cab}]_{(e, \alpha, \beta)(f, \sigma, \delta)} [R_d^{cf}]_{\delta\psi} [F_d^{abc}]_{(f, \sigma, \psi)(g, \mu, \nu)} \quad (4.61)$$

and

$$\sum_{\lambda, \gamma} [(R_e^{ca})^{-1}]_{\alpha\lambda} [F_d^{acb}]_{(e, \lambda, \beta)(g, \gamma, \nu)} [(R_g^{cb})^{-1}]_{\gamma\mu} \quad (4.62)$$

$$= \sum_{f, \sigma, \delta, \psi} [F_d^{cab}]_{(e, \alpha, \beta)(f, \sigma, \delta)} [(R_d^{cf})^{-1}]_{\delta\psi} [F_d^{abc}]_{(f, \sigma, \psi)(g, \mu, \nu)} \quad (4.63)$$

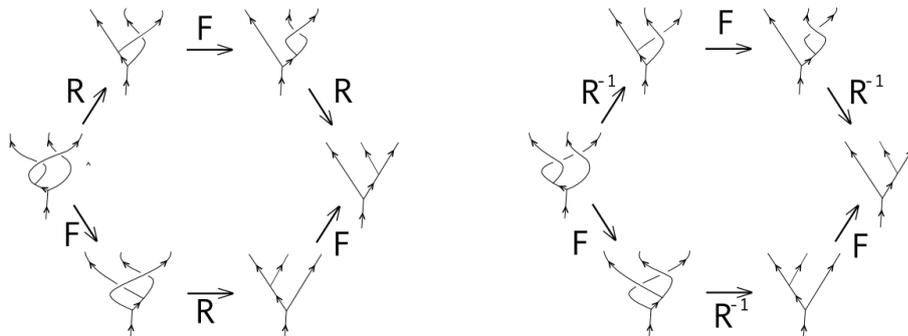
The pentagon and hexagon equations first appeared in this form in the physics literature in [62].

### 4.4 Modularity

The consistent definition fusion rules and unitary  $F$ -symbols and  $R$ -symbols gives a unitary braided tensor category. But in relation to topological quantum field theories, models obeying an additional condition are of particular interest. These are known as modular tensor categories. The characteristic property is that the topological  $S$ -matrix is non-degenerate.

The topological  $S$ -matrix, which is defined by

$$S_{ab} = \frac{1}{\mathcal{D}} \left( \begin{array}{c} \text{Diagram of two circles labeled } a \text{ and } b \text{ with arrows} \end{array} \right) \quad (4.64)$$



**Figure 4.2: The hexagon relations** - The hexagon relations relate fusion to braiding. They imply the Yang-Baxter equation.

encodes a wealth of information about the theory. By applying the monodromy equation, we find

$$S_{ab} = \frac{1}{\mathcal{D}} \sum_c N_{ab}^c \frac{\theta_c}{\theta_a \theta_b} d_c \quad (4.65)$$

From this expression, we derive that  $S_{ab} = S_{ba} = S_{\bar{a}\bar{b}}^*$  and  $S_{0a} = \frac{1}{\mathcal{D}} d_a$ . A very useful equality is

$$a \begin{array}{c} \uparrow \\ \circlearrowleft \\ \downarrow \\ b \end{array} = \frac{S_{ab}}{S_{0b}} \begin{array}{c} \uparrow \\ | \\ b \end{array} \quad (4.66)$$

This holds, because both left and right side are elements of the one-dimensional vector space  $V_b^b$ , hence must be proportional. The constant of proportionality is checked by taking the quantum trace. This is a convenient trick to prove equalities of single-anyon operators. Often, these immediately imply a corresponding result for multi-anyon operators by decomposing these using (4.25).

Modular tensor categories are defined by the non-degeneracy of the  $S$ -matrix. When the theory is unitary, as is the case for physical theories, the  $S$ -matrix is in fact a unitary matrix. Together with the matrix  $T$  with coefficients

$$T_{ab} = e^{2\pi c/24} \theta_a \delta_{ab} \quad (4.67)$$

it forms a representation of the modular group  $SL(2, \mathbb{Z})$  with defining relations  $(ST)^3 = S^2$  and  $S^4 = 1$ . Here  $c$  is the topological central charge, which is equal to the CFT central charge mod 24 when applicable. It can be determined mod 8 from the twist factors and quantum dimensions by the relation

$$\exp\left(\frac{2\pi i c}{8}\right) = \frac{1}{\mathcal{D}} \sum_a d_a^2 \theta_a \quad (4.68)$$

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This is all well known, but since it is an interesting application of the diagrammatic formalism we include a derivation based on [14] of the relations between  $S$  and  $T$  below.

Let

$$p^\pm = \sum_a d_a^2 \theta_a^{\pm 1} \quad (4.69)$$

The following equations hold

$$\begin{aligned} \omega \begin{array}{c} \uparrow \\ \curvearrowright \\ | \\ b \end{array} &= p^+ \theta_b^{-1} \begin{array}{c} | \\ | \\ | \\ b \end{array}, & \omega \begin{array}{c} \uparrow \\ \curvearrowleft \\ | \\ b \end{array} &= p^- \theta_b \begin{array}{c} | \\ | \\ | \\ b \end{array} \end{aligned} \quad (4.70)$$

To show these equalities, note that the left and right hand side of these equations must be proportional, since they are both single-anyon operators. Hence we take the quantum trace on either side to check the proportionality constant. For the equation involving  $p^+$ , the quantum trace of the right hand side is evidently  $p^+ d_b / \theta_b$ , while the left hand side gives

$$\sum_a d_a \theta_a \mathcal{D} S_{ab} = \sum_{a,c} N_{ab}^c d_a d_c \theta_c / \theta_b \quad (4.71)$$

$$= \sum_c d_c^2 \theta_c d_b / \theta_b \quad (4.72)$$

$$= p^+ d_b / \theta_b \quad (4.73)$$

Here we used that  $\sum_a N_{ab}^c d_a = \sum_a N_{cb}^a d_a = d_b d_c$  by (4.38), the symmetries of the fusion coefficients, and the fact that  $d_a = d_{\bar{a}}$ . As a consequence we get

$$\omega \begin{array}{c} \uparrow \uparrow \\ \curvearrowright \\ | \quad | \\ a \quad b \end{array} = p^+ \theta_a^{-1} \theta_b^{-1} \begin{array}{c} \uparrow \uparrow \\ \curvearrowleft \\ | \quad | \\ a \quad b \end{array} \quad (4.74)$$

To see this, apply the ribbon property on the double clockwise braid on the right hand side, i.e. on the operator  $(R_{ab}^\dagger)^2$ , and compare with (4.70).

Define

$$T'_{ab} = \theta_a \delta_{ab}, \quad C_{ab} = \delta_{a\bar{b}} \quad (4.75)$$

Thus  $T'$  is just  $T$  with the phase in front left out, and  $C$  is the charge conjugation matrix. Clearly,  $C$  commutes with  $S$  and with  $T'$ , and  $C^2 = 1$ . But more interestingly

1.  $(ST')^3 = p^+ S^2 C / \mathcal{D}$
2.  $(ST'^{-1})^3 = p^- S^2 / \mathcal{D}$
3.  $S^2 = p^+ p^- C / \mathcal{D}^2$



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This third statement has an important consequence. It implies that loops labelled by  $\omega$  have the killing property. This somewhat hostile name comes from the fact that all non-trivial charge lines going through a loop labelled by  $\omega = \sum_a d_a a$  get killed in the following way.

$$\omega \begin{array}{c} \uparrow \\ \circlearrowleft \\ \downarrow \\ a \end{array} = p^+ p^- \delta_{0a} \begin{array}{c} \uparrow \\ | \\ a \end{array} \quad (4.85)$$

This is again proved by comparing the quantum trace of the left and right hand side. The trace of the left hand side gives

$$\mathcal{D} \sum_b d_b S_{ba} = \mathcal{D}^2 \sum_b S_{0b} S_{ba} = p^+ p^- C_{0a} = p^+ p^- \delta_{0a} \quad (4.86)$$

which is clearly the same as the trace of the right hand side. Thus loops labelled by  $\omega$  give a natural projector in the theory.<sup>1</sup> Taking  $a = 0$  we get the following result

$$p^+ p^- = \mathcal{D}^2 \quad (4.87)$$

If we now define

$$T = \zeta^{-1} T', \quad \zeta = (p^+ / p^-)^{\frac{1}{6}} \quad (4.88)$$

we find that  $S$  and  $T$  obey the relations of the modular group, as was alluded to before. This also gives the implicit definition of the topological central charge  $c$  in terms of the quantum dimensions and twist factors of the theory.

### 4.4.1 Verlinde formula

The famous Verlinde formula, which first appeared in the context of CFT in [76], expresses the fusion coefficients in terms of the  $S$ -matrix as

$$N_{ab}^c = \sum_x \frac{S_{ax} S_{bx} S_{\bar{c}x}}{S_{0x}} \quad (4.89)$$

Because the fusion matrices by associativity of fusion, a basis of common eigenvectors is expected. The Verlinde formula says that this basis is essentially given by the columns of the  $S$ -matrix.

We will now prove the Verlinde formula using the graphical formalism. Define

$$\Lambda_{bc}^{(a)} = \lambda_b^{(a)} \delta_{bc}, \quad \lambda_b^{(a)} = \frac{S_{ab}}{S_{0b}} \quad (4.90)$$

Recall that the fusion matrices are given by  $(N_a)_{bc} = N_{ab}^c$ . We will show that

$$N_a S = S \Lambda^{(a)} \quad (4.91)$$

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<sup>1</sup>This projector has recently been used to provide a spacetime picture for the Levin-Wen models in an interesting work [23]





Let us end this chapter by discussing the full braid operator  $B$  or  $B$ -move. It is defined by

$$\begin{array}{c} a \swarrow \\ \quad b \uparrow \\ \quad \quad c \searrow \\ \quad \quad \quad \downarrow e \\ \quad \quad \quad \quad \downarrow d \end{array} = \sum_f [B_d^{abc}]_{ef} \begin{array}{c} a \swarrow \\ \quad b \uparrow \\ \quad \quad c \searrow \\ \quad \quad \quad \downarrow f \\ \quad \quad \quad \quad \downarrow d \end{array} \quad (4.104)$$

Reinstalling the Greek indices, we find that

$$[B_d^{abc}]_{(e,\alpha,\beta)(f,\mu,\nu)} = \sum_{g,\gamma,\delta,\eta} [F_d^{acb}]_{(e,\alpha,\beta)(g,\gamma,\delta)} [R_g^{cb}]_{\gamma\eta} [(F_d^{abc})^{-1}]_{(g,\eta,\delta)(f,\mu,\nu)} \quad (4.105)$$

This operator captures the effect of braiding on general anyonic states. The monodromy matrix is for example recovered as the  $(0, 0)$  of the appropriate square of this operator,

$$M_{ab} = \sum_{f,\mu,\nu} [B_b^{\bar{a}ba}]_{0,(f,\mu,\nu)} [B_b^{\bar{a}ba}]_{(f,\mu,\nu),0} \quad (4.106)$$

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## Examples of anyon models

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In principle, the unitary braided tensor categories discussed in chapter 4 can be constructed in a three step procedure: First, fix a set of particle labels and fusion rules. Next, solve the pentagon relation to find a consistent set of  $F$ -symbols. Finally, solve the hexagon relations to find  $R$ -symbols.

If no solution to the pentagon relations exists, the declared fusion rules are inconsistent with local quantum mechanics and no model with these fusion rules exists. When we do find a solution to the pentagon relations, but not to the hexagon relations, there is no way to define a consistent braiding for the anyons which excludes the model with these fusion rules as an effective theory for bulk anyons but, as we will see, might describe boundary effects of a medium carrying anyons. If there are multiple solutions for the pentagon/hexagon relations that are not related by a gauge transformation, there are apparently non-equivalent anyon models that obey the same fusion rules. A theorem known as Ocneanu rigidity ensures that, for a given set of fusion rules, there can only be a finite number of non-equivalent anyon models.

Although the procedure outlined above can in principle be used to find all anyon models, in practice it is not feasible to find many interesting models this way. The reason is that the number of equations in the pentagon relations, as well as the number of variables, grows very rapidly with the number of charges in the theory. For small numbers of charges, some classification results have been obtained by direct attack of the pentagon/hexagon relations. For example, all models with up to four charges were classified in [71]. Also, in [22], solutions to the pentagon and hexagon relations were found explicitly for a number of interesting fusion rules by solving them directly.

Most known models, however, were constructed via a completely different route, namely as representation categories of quantum group that we discussed before. In this chapter we will show how to obtain the fusion rules,  $F$ -symbols and  $R$ -symbols for the quantum double. We will also give this data for the  $\mathfrak{su}(2)_k$  theories related to

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$U_q[\mathfrak{su}(2)]$ , Chern-Simons theory and the WZW-models and we flesh out the details for some representative examples that will be used later. But we will start by discussing the well know Fibonacci anyons, for which the pentagon and hexagon equations can be solved by hand.

### 5.1 Fibonacci

Suppose we have only one non-trivial charge besides the vacuum, which we label by 1. The two possibilities for the one non-trivial fusion rule are  $1 \times 1 = 0$  and  $1 \times 1 = 0 + 1$ . The first gives an Abelian theory that is not very interesting, so we take the second rule. This fixes the theory completely, as we will see, and gives rise to the so-called Fibonacci anyons. The discussion below is based on [66].

The  $F$ -symbols we need to find are  $[F_0^{111}]_{ef}$  and  $[F_1^{111}]_{ef}$ . The matrix  $[F_0^{111}]_{ef}$ , has only one non-zero component, which we can set to unity. This gives

$$[F_0^{111}]_{ef} = \delta_{e1}\delta_{f1} \quad (5.1)$$

This leaves us with the task to find the  $2 \times 2$  matrix  $F$  with  $F_{ef} = [F_0^{111}]_{ef}$ . Since we are looking for a unitary theory, we search an  $F$  of the form

$$F = \begin{pmatrix} z & w \\ w^* & z \end{pmatrix} \quad (5.2)$$

with  $|z|^2 + |w|^2 = 1$ . Without multiplicities in the theory, the pentagon relations simplify to

$$[F_e^{fcd}]_{gl}[F_e^{abl}]_{fk} = \sum_h [F_g^{abc}]_{fh}[F_e^{ahd}]_{gk}[F_k^{bcd}]_{hl} \quad (5.3)$$

Now take  $a = b = c = d = e = f = k = 1$  and  $g = l = 0$ . The pentagon equation for this combination of labels is

$$[F_1^{111}]_{00} = [F_1^{111}]_{01}[F_1^{111}]_{10} \quad (5.4)$$

which gives  $z = |w|^2$ . This leads us to the general solution

$$F = \begin{pmatrix} \phi^{-1} & e^{i\theta}\sqrt{\phi^{-1}} \\ e^{-i\theta}\sqrt{\phi^{-1}} & -\phi^{-1} \end{pmatrix} \quad (5.5)$$

where  $\phi = \frac{1+\sqrt{5}}{2}$ , i.e the golden ratio. The phase  $e^{i\theta}$  can be set to unity with a gauge transformation. All pentagon relations are now indeed satisfied.

Next we will solve the hexagon relations. Without multiplicities, these take the following form.

$$R_e^{ac}[F_d^{acb}]_{eg}R_g^{bc} = \sum_f [F_d^{cab}]_{ef}R_d^{fc}[F_d^{abc}]_{fg} \quad (5.6)$$

$$(R_e^{ac})^{-1}[F_d^{acb}]_{eg}(R_g^{bc})^{-1} = \sum_f [F_d^{cab}]_{ef}(R_d^{fc})^{-1}[F_d^{abc}]_{fg} \quad (5.7)$$

Since the vacuum braids trivially, we only need to find the  $R$ -symbols with the two upper indices equal to 1, so only  $R_0 = R_0^{11}$  and  $R_1 = R_1^{11}$  corresponding to a pair of Fibonacci anyons having trivial or non-trivial total charge respectively. We find

$$R_0 = e^{-4\pi i/5}, \quad R_1 = e^{3\pi i/5} \quad (5.8)$$

We can read off the quantum dimension  $d_1 = \frac{1+\sqrt{5}}{2}$  from  $F$ , and find that then non-trivial particle has spin  $\theta_1 = \mathbf{fb}_1(R_0^{11})^* = e^{2\pi i \frac{2}{5}}$ .

Now let  $N_0^n = N_0^{1 \times n}$  be the dimension of the topological Hilbert space of  $n$  Fibonacci anyons with trivial total charge. It is easy to see that

$$N_0^1 = 0, \quad N_0^2 = 1, \quad N_0^3 = 1, \quad N_0^4 = 2, \quad N_0^5 = 3 \quad (5.9)$$

when looking at the number of fusion states explicitly. In general, the  $N_0^n$  obey a simple recursion relation. If the first two particles have trivial total charge, the  $n - 2$  other particles can fuse in  $N_0^{n-2}$  different ways. When the charge of the first two particles is 1, the other  $n - 2$  charges can combine with this charge in  $N_0^{n-1}$  ways. So the multiplicities obey the recursion relation

$$N_0^n = N_0^{n-1} + N_0^{n-2} \quad (5.10)$$

which is well known to produce the Fibonacci numbers. Indeed, one recognises the first of these in the list 5.9 This is why these anyons are given their specific name.

## 5.2 Quantum double

Using the facts about the quantum double  $D[H]$  and its representations from chapter 3, it is possible to deduce fusion rules,  $F$ -symbols and  $R$ -symbols, and thus to obtain a description of the fusion braiding properties of the excitations in discrete gauge theories in the language of the previous chapter.

The connection with the graphical formalism and the representations of the quantum double is as follows. Diagrams correspond to operators that commute with the action of  $D[H]$  in the internal space, i.e. operators that are gauge invariant operators and commute with flux projections. By Schurs lemma, these must be proportional to the identity for irreps.

The identity operator of the carrier space  $V_\alpha^A$  is represented by a charge line labelled by  $a = (A, \alpha)$  directed upwards. The splitting vertex labelled by  $a = (A, \alpha)$ ,  $b = (B, \beta)$  on top and  $c = (C, \gamma)$  on the bottom is identified with the inclusion of  $V_\gamma^C$  in  $V_\alpha^A \otimes V_\beta^B$ , up to a scale factor. The dual fusion vertex, on the other hand, is associated with the projection onto  $V_\gamma^C$  as irreducible subspace of  $V_\alpha^A \otimes V_\beta^B$ . If there are multiple copies of  $V_\gamma^C$  present in  $V_\alpha^A \otimes V_\beta^B$ , we can label them with indices  $\mu$ . Note that a unitary transformation on the  $\mu$ 's in  $\oplus_\mu (V_\gamma^C)_\mu$  leaves the representations in tact. The quantum dimension and quantum trace correspond to the usual dimension and trace

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of the modules  $V_\alpha^A$ .

$$\begin{array}{c} \uparrow \\ a \end{array} \text{ identity,} \quad \begin{array}{c} a \quad b \\ \quad \mu \\ \quad \uparrow \\ c \end{array} \text{ inclusion,} \quad \begin{array}{c} c \\ \quad \mu \\ \quad \uparrow \\ a \quad b \end{array} \text{ projection} \quad (5.11)$$

Throughout this section, it will be very convenient to condense the notation, as the discussion above already shows. We agree to write  $a, b, c, \dots$  for the particle-labels  $(A, \alpha), (B, \beta), (C, \gamma), \dots$ . To avoid confusing the internal space  $V_\alpha^A$  with the fusion/splitting spaces, we will also refer to it as  $a$ , and similar for the other particle labels. The representation maps are denoted  $\Pi^a = \Pi_\alpha^A$ , etcetera. We can label the basis states of  $a$  by indices  $i = 1, \dots, d_a$ . We denote the basis states of  $a$  therefore as

$$\{|a, i\rangle\}, \quad i = 1, \dots, d_a \quad (5.12)$$

leaving the particle label explicit, in stead of  $|h_j, {}^\alpha v_k\rangle$  as we did before. The coordinate functions of the representation matrices are denoted  $\Pi_{ij}^a$ . The fusion product  $a \times b$  means the tensor product representation.

### 5.2.1 Fusion rules

The fusion coefficients  $N_{ab}^c$ , corresponding to the decomposition

$$a \times b = \Pi^a \otimes \Pi^b = \bigoplus_c \Pi^c \sum_c N_{ab}^c c \quad (5.13)$$

can be obtained by using certain orthogonality relations for the characters of representation. These are well known in the case of groups, but generalize to representations of the quantum double. It means that for unitary representations, we have

$$\frac{1}{|H|} \sum_{h,g} \text{Tr}(\Pi^a(P_h g)) \text{Tr}(\Pi^b(P_h g)^*) = \delta_{ab} \quad (5.14)$$

In fact, more general orthogonality relations for the coordinate functions  $\mathfrak{P}_{ij}^a$ . They obey

$$\sum_{h,g \in H} \Pi_{ij}^a(P_h g)^* \Pi_{kl}^b(P_h g) = \delta_{a,b} \delta_{i,k} \delta_{j,l} \quad (5.15)$$

This follows from Woronowicz's theory of pseudogroups [85] and generalize the well known Schur orthogonality relations for groups. We will use them heavily in the calculation of  $F$ -symbols.

From the orthogonality of characters, we see that the fusion multiplicities may be obtained from the representations matrices as

$$N_{ab}^c = \frac{1}{|H|} \sum_{h,g} \text{Tr}(\Pi^a \otimes \Pi^b(\Delta(P_h g))) \text{Tr}(\Pi^c(P_h g)^*) \quad (5.16)$$

$$= \sum_{h,g} \sum_{h'h''=h} \left( \sum_{i,j} \Pi_{ij}^a(P_{h'} g) \Pi_{ij}^b(P_{h''} g) \right) \left( \sum_k \Pi_{kk}^c(P_h g) \right) \quad (5.17)$$

Next, we show how to calculate  $F$ -symbols.



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as a consequence of the orthogonality relation (5.15). Applying the projector to a direct product of two states and using equation (5.22), gives

$$\begin{aligned} \mathcal{P}_{lk}^c |a, i\rangle |b, j\rangle &= \begin{pmatrix} c & a & b \\ k & i & j \end{pmatrix} |c, l\rangle \\ &= \sum_{i', j'} \begin{pmatrix} c & a & b \\ k & i & j \end{pmatrix} \begin{pmatrix} a & b & c \\ i' & j' & l \end{pmatrix} |a, i'\rangle |b, j'\rangle. \end{aligned} \quad (5.25)$$

By using the definition (5.23), this is seen to be equal to

$$\begin{aligned} \mathcal{P}_{lk}^c |a, i\rangle |b, j\rangle &= \frac{d_c}{|H|} \sum_{h, g \in H} \Pi_{lk}^{c*}(P_h g) \Pi(P_h g) |\Pi_\alpha^A, i\rangle |\Pi_\beta^B, j\rangle \\ &= \frac{d_c}{|H|} \sum_{h, g \in H} \Pi_{lk}^{c*}(P_h g) \sum_{h' h''=h} \Pi_{i'i}^a(P_{h'} g) \Pi_{j'j}^b(P_{h''} g) |a, i'\rangle |b, j'\rangle, \end{aligned} \quad (5.26)$$

where we have inserted the definition of the comultiplication to act on the product state.

Equating expressions (5.25) and (5.26), we obtain

$$\begin{pmatrix} c & a & b \\ k & i & j \end{pmatrix} \begin{pmatrix} a & b & c \\ i' & j' & l \end{pmatrix} = \frac{d_c}{|H|} \sum_{h, g \in H} \Pi_{lk}^{c*}(P_h g) \sum_{h' h''=h} \Pi_{i'i}^a(P_{h'} g) \Pi_{\beta j' j}^B(P_{h''} g) \quad (5.27)$$

Unitarity amounts to

$$\begin{pmatrix} c & a & b \\ k & i & j \end{pmatrix} = \begin{pmatrix} a & b & c \\ i & j & k \end{pmatrix}^*. \quad (5.28)$$

Now pick some triple  $(i, j, k)$  such that

$$\frac{d_c}{|H|} \sum_{h, g \in H} \Pi_{kk}^{c*}(P_h g) \sum_{h' h''=h} \Pi_{ii}^a(P_{h'} g) \Pi_{jj}^b(P_{h''} g) \quad (5.29)$$

is non-zero. From equation (5.27) with  $i = i'$ ,  $j = j'$  and  $k = k'$  and from the unitarity condition (5.28) it follows that this number is real and positive. This fixes one of the Clebsch-Gordan symbols, by

$$\begin{pmatrix} a & b & c \\ i & j & k \end{pmatrix} = \left( \frac{d_c}{|H|} \sum_{h, g \in H} \sum_{h' h''=h} \Pi_{kk}^{c*}(P_h g) \Pi_{ii}^a(P_{h'} g) \Pi_{jj}^b(P_{h''} g) \right)^{1/2} \quad (5.30)$$

We can use equation (5.27) to calculate all others, which results in

$$\begin{pmatrix} a & b & c \\ i' & j' & k' \end{pmatrix} = \sqrt{\frac{d_c}{|H|}} \frac{\sum_{h, g \in H} \sum_{h' h''=h} \Pi_{k'k}^{c*}(P_h g) \Pi_{i'i}^a(P_{h'} g) \Pi_{j'j}^b(P_{h''} g)}{\left( \sum_{h, g \in H} \sum_{h' h''=h} \Pi_{kk}^{c*}(P_h g) \Pi_{ii}^a(P_{h'} g) \Pi_{jj}^b(P_{h''} g) \right)^{1/2}}$$

From the Clebsch-Gordan symbols it is straightforward to calculate the  $F$ -symbols. The definitions imply the relation

$$\sum_{f;n} [F_d^{abc}]_{e,f} \begin{pmatrix} b & c & f \\ j & k & n \end{pmatrix} \begin{pmatrix} a & f & d \\ m & k & \ell \end{pmatrix} = \begin{pmatrix} a & b & e \\ i & j & m \end{pmatrix} \begin{pmatrix} e & c & d \\ m & k & \ell \end{pmatrix} \quad (5.31)$$

It follows from (5.28) that

$$\sum_{i,j} \begin{pmatrix} a & b & c \\ i & j & k \end{pmatrix} \begin{pmatrix} a & b & c' \\ i & j & k' \end{pmatrix}^* = \delta_{cc'} \delta_{kk'} \quad (5.32)$$

Together, these give the following expression for the  $F$ -symbols in terms of the Clebsch-Gordan coefficients

$$[F_d^{abc}]_{e,f} = \sum_{i,j,k,m,n} \begin{pmatrix} b & c & f \\ j & k & n \end{pmatrix} \begin{pmatrix} a & f & d \\ m & k & \ell \end{pmatrix} \begin{pmatrix} a & b & e \\ i & j & m \end{pmatrix}^* \begin{pmatrix} e & c & d \\ m & k & \ell \end{pmatrix}^* \quad (5.33)$$

(Note that  $\ell$  can be chosen freely.)

In principle, one can now run a computer program to calculate the Clebsch-Gordan coefficients and subsequently the  $F$ -symbols from the above description.

### 5.2.3 Braiding

Recall that the braiding for the quantum double is defined through the action of the universal  $R$ -matrix

$$\mathcal{R} = \sum_{h,g \in H} P_g e \otimes P_h g \quad (5.34)$$

and subsequent flipping of the tensor legs. Because  $\Delta(P_h g)\mathcal{R} = \mathcal{R}\Delta(P_h g)$ , the effect of braiding on irreducible subspaces  $c$  of  $a \times b \simeq b \times a$  is in principle just multiplication by a complex number  $R_c^{ab}$  by Schur's lemma. But if some  $c$  occurs in  $a \times b$  with multiplicity greater than one, it can in fact act as a unitary transformation  $[R_c^{ab}]_{\mu\nu}$  in the subspace of total charge  $c$ .

We can derive the  $R$ -symbols by composing the braiding with the inclusion of  $c$ . Denote the representation matrix of the action of  $\mathcal{R}$  on a state  $|a, i\rangle |b, j\rangle$  as  $\mathcal{R}_{(ij), (i'j')}^{a,b}$ , i.e.

$$\mathcal{R}_{(ij), (i'j')}^{a,b} \equiv \Pi_{i'j'}^a \otimes \Pi_{ij}^b(\mathcal{R}) \quad (5.35)$$

For the simple case where there are no multiplicities of  $c$ , we get

$$R_c^{ab} = \sum_{i,i'=1}^{d_a} \sum_{j,j'=1}^{d_b} \begin{pmatrix} a & b & c \\ i & j & \ell \end{pmatrix} \begin{pmatrix} b & a & c \\ j' & i' & \ell \end{pmatrix}^* \mathcal{R}_{(ij), (i'j')}^{a,b} \quad (5.36)$$

where again  $\ell$  can be chosen freely.

If multiplicities of  $c$  do occur, we can still obtain the  $[R_c^{ab}]_{\mu\nu}$  by first including the  $\mu^{\text{th}}$  copy of  $c$  and next project onto the  $\nu^{\text{th}}$  copy. This can be done by introducing more

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general Clebsch-Gordan coefficients that also carry an index  $\mu$  that keeps track of the copy of  $c$ . The formula then becomes

$$[R_c^{ab}]_{\mu\nu} = \sum_{i,i'=1}^{d_a} \sum_{j,j'=1}^{d_b} \begin{pmatrix} a & b & | & c, \mu \\ i & j & | & \ell \end{pmatrix} \begin{pmatrix} c, \nu & | & b & a \\ \ell & | & j' & i' \end{pmatrix} \mathcal{R}_{(ij),(i'j')}^{a,b} \quad (5.37)$$

### 5.3 The $\mathfrak{su}(2)_k$ theories

The data for  $\mathfrak{su}(2)_k$  theories, which were given this name because they appear in the WZW-models based on the affine algebra of  $\mathfrak{su}(2)$  at level  $k$ , can be derived from the representation theory of  $U_q[\mathfrak{su}(2)]$  where  $q$  is a root of unity and related to  $k$  by as  $q = e^{i\frac{2\pi}{k+2}}$ . As for general values of  $q$ , a similar structure for the representations appears as in the representation theory of  $\mathfrak{su}(2)$ , with integers replaced by the  $q$ -numbers  $[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$ . But, when considering the appropriate category of representations, the highest values for the highest weights get truncated. There are only a finite number of irreducible representations corresponding to the finite set of charge labels. We will state the general formulas as they appeared in [72], where the authors refer to [47] for the calculation.

The charges of  $\mathfrak{su}(2)_k$  are labelled by integers  $a = 0, 1, \dots, k$  that are related to the highest weights  $j$  as  $a = 2j$  (a notation also frequently used). The fusion rules are given by a truncated version of the usual addition rules for  $SU(2)$ -spin

$$a \times b = \sum_c N_{ab}^c c \quad (5.38)$$

$$= |a - b| + (|a - b| + 2) + \dots + \min\{a + b, 2k - a - b\} \quad (5.39)$$

i.e.  $N_{ab}^c = 1$  when  $|a - b| \leq c \leq \min\{a + b, 2k - a - b\}$  and  $a + b + c = 0 \pmod{2}$ , and zero otherwise.

For the  $F$ -symbols, one has the general formula

$$[F_d^{abc}]_{ef} = (-1)^{(a+b+c+d)/2} \sqrt{[e+1]_q [f+2]_q} \left\{ \begin{array}{cc|c} a & b & e \\ c & d & f \end{array} \right\} \quad (5.40)$$

where

$$\left\{ \begin{array}{cc|c} a & b & e \\ c & d & f \end{array} \right\} = \Delta(a, b, e) \Delta(e, c, d) \Delta(b, c, f) \Delta(a, f, d) \\ \times \sum_z \left\{ \frac{(-1)^z [z+1]_q!}{[z - \frac{a+b+e}{2}]_q! [z - \frac{e+c+d}{2}]_q! [z - \frac{b+c+f}{2}]_q! [z - \frac{a+f+d}{2}]_q!} \right. \\ \left. \times \frac{1}{[\frac{a+b+c+d}{2} - z]_q! [\frac{a+e+c+f}{2} - z]_q! [\frac{b+c+d+f}{2} - z]_q!} \right\} \quad (5.41)$$

with

$$\Delta(a, b, c) = \sqrt{\frac{[ \frac{-a+b+c}{2} ]_q! [ \frac{a-b+c}{2} ]_q! [ \frac{a+b-c}{2} ]_q!}{[ \frac{a+b+c}{2} + 1 ]_q}}, \quad \text{and} \quad [n]_q! = \prod_{m=1}^n [m]_q \quad (5.42)$$

The sum over  $z$  should run over all integers for which the  $q$ -factorials are well defined, i.e. such that no of the arguments become less than zero. This depends on the level  $k$ . The expression for  $\Delta$  is only well defined for admissible triples  $(a, b, c)$ , by which we mean that  $a + b + c = 0 \pmod{2}$  and  $|a - b| \leq c \leq a + b$  (we will take it to be zero for other triples, implementing consistency with the fusion rules). Note that  $\Delta$  is invariant under permutations of the input.

The  $R$ -symbols are given by the general equation

$$R_c^{ab} = (-1)^{c-a-b} q^{\frac{1}{8}(c(c+2)-a(a+2)-b(b+2))} \quad (5.43)$$

This gives topological spins

$$\theta_a = e^{2\pi i \frac{a(a+2)}{4(k+2)}} \quad (5.44)$$

The quantum dimensions are

$$d_a = \frac{\sin\left(\frac{(a+1)\pi}{k+2}\right)}{\sin\left(\frac{\pi}{k+2}\right)} \quad (5.45)$$

For later reference, we will work out the values of these functions for some representative values of  $k$ .

**k=2** The simplest non-trivial case occurs for  $k = 2$  which is closely related to the Ising model. The charges are denoted  $0, 1, 2$ , with non-trivial fusion rules

$$1 \times 1 = 0 + 2, \quad 1 \times 2 = 1, \quad 2 \times 2 = 0 \quad (5.46)$$

The relevant  $q$ -numbers and factorials are

$n$	0	1	2	3
$[n]_q$	0	1	$\sqrt{2}$	1
$[n]_q!$	1	1	$\sqrt{2}$	$\sqrt{2}$

(5.47)

This gives quantum dimensions and spins

$\mathfrak{su}(2)_2$		
0	$d_0 = 1$	$h_0 = 0$
1	$d_1 = 2$	$h_1 = \frac{3}{16}$
2	$d_2 = 1$	$h_2 = \frac{1}{2}$

(5.48)

Using the general formula, we can calculate the  $F$ -symbols. Let us calculate  $[F_1^{111}]_{00}$ . The relevant  $\Delta$ 's are

$$\Delta(1, 1, 0) = \Delta(1, 0, 1) = \Delta(0, 1, 1) = \sqrt{\frac{[0]_q! [0]_q! [1]_q!}{[2]_q!}} = \frac{1}{2^{1/4}} \quad (5.49)$$

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This gives

$$\begin{aligned}
 \left\{ \begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array} \right\} &= \frac{1}{2} \sum_z \frac{(-1)^z [z+1]_q!}{([z-1]_q!)^4 [2-z]_q! [1-z]_q!} \\
 &= \frac{1}{2} \frac{-[2]_q!}{([0]_q!)^4 [1]_q! [0]_q!} \\
 &= -1/\sqrt{2}
 \end{aligned} \tag{5.50}$$

which produces

$$[F_1^{111}]_{00} = -1/\sqrt{2} \tag{5.51}$$

By similar calculations we find

$$[F_1^{111}]_{ef} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad e, f = 0, 2 \tag{5.52}$$

**k=4** For  $\mathfrak{su}(2)_4$  has been a very important example while working on this thesis. Since the primary goal has been to study Bose condensation, it is of particular interest because it is one of the most simple models with a non-trivial boson.

The charges of  $\mathfrak{su}(2)_4$  are labelled by 0, 1, 2, 4 with the fusion rules given by the general formula, for example

$$2 \times 2 = 0 + 2 + 4, \quad 1 \times 3 = 2 + 4 \tag{5.53}$$

The relevant  $q$ -numbers and  $q$ -factorials are

$n$	0	1	2	3	4	5	(5.54)
$[n]_q$	0	1	$\sqrt{3}$	2	$\sqrt{3}$	1	
$[n]_q!$	1	1	$\sqrt{3}$	$2\sqrt{3}$	6	6	

This gives spins and quantum dimensions

$\mathfrak{su}(2)_4$			(5.55)
0	$d_0 = 1$	$h_0 = 0$	
1	$d_1 = \sqrt{3}$	$h_1 = \frac{1}{8}$	
2	$d_2 = 2$	$h_2 = \frac{1}{3}$	
3	$d_3 = \sqrt{3}$	$h_3 = \frac{5}{8}$	
4	$d_4 = 1$	$h_4 = 1$	

In chapter 6 we need some specific values for the  $F$ -symbols, hence we calculate these here. These are  $[F_4^{242}]_{22}$ ,  $[F_0^{242}]_{22}$ ,  $[F_2^{242}]_{22}$ , or actually  $[F_{22}^{22}]_{40}$ ,  $[F_{22}^{22}]_{42}$  and  $[F_{22}^{22}]_{44}$ . The relevant  $\Delta$ 's are

$$\Delta(2, 2, 2) = 1/\sqrt{6} \quad \Delta(2, 2, 0) = 1/\sqrt{2}, \quad \Delta(2, 2, 4) = 1/\sqrt{2} \tag{5.56}$$

We get

$$[F_4^{242}]_{22} = 2\Delta(2, 4, 2)\Delta(2, 2, 4)\Delta(4, 2, 2)\Delta(2, 2, 4) \quad (5.57)$$

$$\times \sum_z \frac{(-1)^z [z+1]_q!}{([z-4]_q!)^4 [6-z]_q! [4-z]_q! [6-z]_q!} \quad (5.58)$$

$$= 2 \times \frac{1}{4} \times \frac{[5]_q!}{([2]_q!)^2} \quad (5.59)$$

$$= 1 \quad (5.60)$$

Similarly,

$$[F_0^{242}]_{22} = 2\Delta(2, 4, 2)\Delta(2, 2, 0)\Delta(4, 2, 2)\Delta(2, 2, 0) \quad (5.61)$$

$$\times \sum_z \frac{(-1)^z [z+1]_q!}{([z-4]_q!)^2 ([z-2]_q!)^2 ([4-z]_q!)^3} \quad (5.62)$$

$$= 2 \times \frac{1}{4} \times \frac{[5]_q!}{([2]_q!)^2} \quad (5.63)$$

$$= 1 \quad (5.64)$$

and finally,

$$[F_2^{242}]_{22} = -2\Delta(2, 4, 2)\Delta(2, 2, 2)\Delta(4, 2, 2)\Delta(2, 2, 2) \quad (5.65)$$

$$\times \sum_z \frac{(-1)^z [z+1]_q!}{([z-4]_q!)^2 ([z-3]_q!)^2 ([5-z]_q!)^2 ([4-z]_q!)} \quad (5.66)$$

$$= -2 \times \frac{1}{12} \times \frac{[5]_q!}{([0]_q!)^3 ([1]_q!)^4} \quad (5.67)$$

$$= -1 \quad (5.68)$$

By

$$[F_{cd}^{ab}]_{(e,\alpha,\beta)(f,\mu,\nu)} = \sqrt{\frac{d_e d_f}{d_a d_d}} [F_f^{ceb}]_{(a,\alpha,\mu)(d,\beta,\nu)}^* \quad (5.69)$$

this gives

$$[F_{22}^{22}]_{40} = \frac{1}{2}, \quad [F_{22}^{22}]_{42} = -\frac{1}{\sqrt{2}}, \quad [F_{22}^{22}]_{44} = \frac{1}{2} \quad (5.70)$$

**k=10** Also  $\mathfrak{su}(2)_k$  was of particular interest to us. While having a reasonably small number of charges – eleven – it is a much more intricate example than  $\mathfrak{su}(2)_4$ . There is a non-trivial bosonic charge, labelled 6 and the result of condensation was known. Hence it presented a fine test case for the theory of subsequent chapters.

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We list the quantum dimensions and spins here.

$\mathfrak{su}(2)_{10}$		
0	$d_0 = 1$	$h_0 = 0$
1	$d_1 = \sqrt{s + \sqrt{3}}$	$h_1 = \frac{1}{16}$
2	$d_2 = 1 + \sqrt{3}$	$h_2 = \frac{1}{6}$
3	$d_3 = \sqrt{2} + \sqrt{2 + \sqrt{3}}$	$h_3 = \frac{5}{16}$
4	$d_4 = 2 + \sqrt{3}$	$h_4 = \frac{1}{2}$
5	$d_5 = 2\sqrt{2 + \sqrt{3}}$	$h_5 = \frac{35}{48}$
6	$d_6 = 2 + \sqrt{3}$	$h_6 = 1$
7	$d_7 = \sqrt{2} + \sqrt{2 + \sqrt{3}}$	$h_7 = \frac{21}{16}$
8	$d_8 = 1 + \sqrt{3}$	$h_8 = \frac{5}{3}$
9	$d_9 = \sqrt{2 + \sqrt{3}}$	$h_9 = \frac{33}{16}$
10	$d_{10} = 1$	$h_{10} = \frac{5}{2}$

(5.71)

The fusion rules and  $F$ -symbols can be calculated using the general formulae. Calculating the  $F$ -symbols by hand becomes very cumbersome, hence we implemented the formulae in Mathematica to do calculations on  $\mathfrak{su}(2)_{10}$  and general  $\mathfrak{su}(2)_k$  models .

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## Bose condensation in topologically ordered phases

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In this chapter, we start the study of phase transitions between topologically ordered phases, which is the main goal of this thesis. The theory developed in the previous chapters is used to extend the formalism for topological symmetry breaking phase transitions. This should allow one to construct a full diagrammatic formalism for topological symmetry breaking, such that arbitrary diagrams for the symmetry-broken phase can be calculated, using data from the original theory only.

We start with a short review of the topological symmetry breaking approach, which is based on quantum group symmetry breaking by a bosonic condensate. The main idea is that a mechanism of symmetry breaking, very similar to spontaneous symmetry breaking known from the ordinary theory of phase transitions, can account for many transitions between topological phases, only the symmetry is given by a quantum group. This was first explored in the context of discrete gauge theories in [5, 8]. Later, the formalism was refined and applied to nematic phases in liquid crystals in [7, 12, 61]. These earlier approaches had the drawback that they were not applicable to treat cases most interesting for the FQHE, where non-integer quantum dimensions occur. In [9], a general scheme was laid out that can also handle these cases.

An important feature of this general scheme is that the explicit description of the quantum group disappears and everything is done on the level of the representations, or particle labels. In mathematical terminology, one might say that the representation category of the quantum group is put centerstage while the quantum group itself is put to the background. It could therefore be expected that there is a connection with the mathematical literature on (modular) tensor categories.

The generalization of the notion of a subgroup to the appropriate category theoretical context is a commutative algebra, as argued in [48]. The results of [9] and [48] are in perfect agreement, although the language is very different in many respects. This motivated the study of the mathematical literature to see what physically meaningful

## 6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

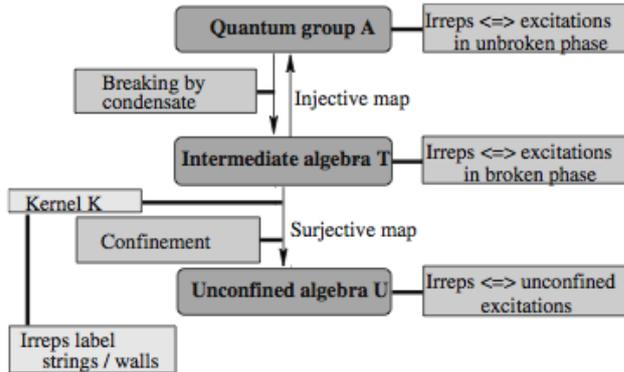
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results could be extracted.

### 6.1 Topological symmetry breaking

We outline the formalism of topological symmetry breaking below. This is described in detail in [9].

The topological symmetry breaking scheme has two steps, or three stages. We start with a (2+1) dimensional system exhibiting topological order with underlying quantum group symmetry labelling the topological sectors. This is the unbroken phase described by the quantum group or theory  $\mathcal{A}$  (we use the notation  $\mathcal{A}$  for both the quantum group and its category of representations, if no confusion is likely to arise).



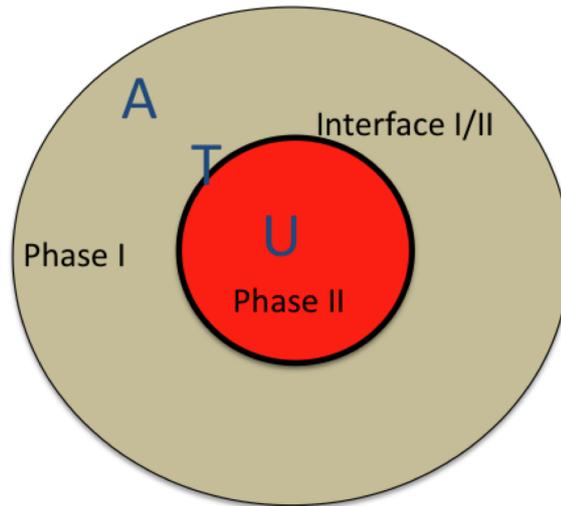
**Figure 6.1: Topological symmetry breaking scheme** - Topological symmetry breaking has two steps, or three stages. The first step is the restriction of the quantum group  $\mathcal{A}$  to the intermediate algebra  $\mathcal{T} \subset \mathcal{A}$ . The second step is the projection onto the Hopf quotient  $\mathcal{U}$  of  $\mathcal{T}$ . Figure from [9].

The first step of the topological symmetry breaking scheme is the formation of a bosonic condensate. For this to happen, the particle spectrum of the  $\mathcal{A}$ -theory should of course include a bosonic excitation. The formation of a condensate reduces the symmetry  $\mathcal{A}$  to some subalgebra  $\mathcal{T}$  of  $\mathcal{A}$  which can be thought of as the stabilizer of the condensate. This  $\mathcal{T}$ -algebra represents the second stage of the symmetry breaking scheme. The precise definition of  $\mathcal{T}$  has been discussed for quantum double theories and generalizations thereof [5, 7, 8, 12, 61]. Since quantum groups are not groups, some subtle issues occur. Which of the structures of  $\mathcal{A}$  must be inherited by  $\mathcal{T}$ ? It is a priori unclear if  $\mathcal{T}$  has the full structure of a quantum group. In fact, allowing the freedom that  $\mathcal{T}$  does not give consistent braiding gives rise to one of the main benefits of the quantum group approach, the description of excitations on the boundary of the broken phase or interface between two phases.

When  $\mathcal{T}$  does not allow for the definition of a universal  $R$ -matrix there is no consistent braiding of the excitations of the  $\mathcal{T}$ -theory. This is caused by excitations that

do not braid trivially with the condensate and thereby locally destroy it. These excitations pull strings in the condensate with energy proportional to their length. In order to minimize the energy, the excitations that pull strings will be confined, either like  $\mathcal{T}$ -hadrons or on the boundary of the system. The intermediate  $\mathcal{T}$ -algebra thus provides a natural description of the boundary excitations. If we think of a geometry where a droplet of the system in the broken phase is surrounded by a region in the unbroken phase, as in figure 6.2, the  $\mathcal{T}$ -theory gives a description of the interface between the two phases.

In the second step of the scheme, we project out the confined excitations to obtain a description of the bulk theory. On the level of the quantum group this corresponds to the surjective Hopf map from  $\mathcal{T}$  to some Hopf quotient  $\mathcal{U}$ . To describe bulk anyons,  $\mathcal{U}$  should give rise to consistent braiding and all other features of the tensor categories we described in chapter 4.



**Figure 6.2: Two-phase system** - The figure a system in the unbroken phase I described by  $\mathcal{A}$  that contains a droplet in the symmetry broken phase II described by  $\mathcal{U}$ . The excitations on I/II interface are naturally described by the intermediate algebra  $\mathcal{T}$ .

### 6.1.1 Particle spectrum and fusion rules of $\mathcal{T}$

Let us describe the formalism in some more detail. To look for a boson  $b$  in the spectrum of  $\mathcal{A}$ , which we label by a Greek index for future convenience, we must first know what we mean by a bosonic excitation in this context. An obvious requirement is trivial spin, expressed by

$$\theta_b = e^{2\pi i h_b} = 1 \quad \longleftrightarrow \quad h_b \in \mathbb{Z} \quad (6.1)$$

But in order to form a stable condensate, there must be a multi-particle state of  $b$ 's that is invariant under braiding. Unlike the higher dimensional case, this is not guaranteed

## 6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

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by the trivial spin condition. Suppose a  $bb$ -pair has total charge  $c$ . The monodromy of the two  $b$ 's leads to a phase  $\exp(2\pi i h_c)$  as follows from

$$\begin{array}{c} b \\ \swarrow \\ \text{---} \\ \searrow \\ b \\ \swarrow \\ \text{---} \\ \searrow \\ c \end{array} = e^{2\pi i(h_c - 2h_b)} \begin{array}{c} b \\ \swarrow \\ \text{---} \\ \searrow \\ b \\ \swarrow \\ \text{---} \\ \searrow \\ c \end{array} = e^{2\pi i h_c} \begin{array}{c} b \\ \swarrow \\ \text{---} \\ \searrow \\ b \\ \swarrow \\ \text{---} \\ \searrow \\ c \end{array} \quad (6.2)$$

Therefore, a second condition is necessary, namely that there is a fusion channel  $c \in b \times b$  with  $h_c \in \mathbb{Z}$ .

One might argue that a full definition should guarantee that there is a fusion channel that is completely invariant under monodromies or even braidings for  $b^{\times n}$  for any  $n$ . This condition is much more involved to check when only the spins and quantum dimensions of the theory are known/used, as is the case in [9]. In the approach that we will develop later, this is in fact guaranteed by the conditions that we impose. This has the draw back that we need the  $F$ -symbols and  $R$ -symbols, which might be harder to obtain.

But for now, a working definition of a boson is

<p>If a charge <math>b \in \mathcal{A}</math> has</p> <ol style="list-style-type: none"> <li>1. Trivial spin, <math>h_b \in \mathbb{Z}</math>,</li> <li>2. A fusion channel <math>c \in b \times b</math> with <math>h_c \in \mathbb{Z}</math>.</li> </ol> <p>then <math>b</math> is a <i>boson</i>.</p>
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We will illustrate the theory for  $\mathfrak{su}(2)_4$ , with fields 0, 1, 2, 3, 4 and non-trivial fusion rules

$$\begin{array}{llll}
 1 \times 1 = 0 + 2 & & & \\
 1 \times 2 = 1 + 3 & 2 \times 2 = 0 + 2 + 4 & & \\
 1 \times 3 = 2 + 4 & 2 \times 3 = 1 + 3 & 3 \times 3 = 0 + 2 & \\
 1 \times 4 = 3 & 2 \times 4 = 2 & 3 \times 4 = 1 & 4 \times 4 = 0
 \end{array} \quad (6.3)$$

For convenience, we list the spins and quantum dimensions again.

$\mathfrak{su}(2)_4$		
0	$d_0 = 1$	$h_0 = 0$
1	$d_1 = \sqrt{3}$	$h_1 = \frac{1}{8}$
2	$d_2 = 2$	$h_2 = \frac{1}{2}$
3	$d_3 = \sqrt{3}$	$h_3 = \frac{3}{8}$
4	$d_4 = 1$	$h_4 = 1$

(6.4)

From this we see that 4 is non-trivial boson, according to the conditions formulated above. It has trivial spin and, because  $4 \times 4 = 0$ , also the second condition is satisfied.

The next step is to work out what happens after condensation. Restricting the symmetry to  $\mathcal{T} \subset \mathcal{A}$  can be noticed on the level of representations by two effects.

## 6.1 Topological symmetry breaking

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Different representations of  $\mathcal{A}$  might be isomorphic when we restrict to the subalgebra  $\mathcal{T}$ , which gives *identifications*. And, representations that were irreducible under the action of  $\mathcal{A}$  might have invariant subspaces under the transformations of  $\mathcal{T}$ , which leads to decomposition into a direct sum of irreducible  $\mathcal{T}$ -modules. This, we call *splitting*. These effects are fully captured by a set of branching rules, or restriction map,

$$a \rightarrow \sum_t n_a^t t \quad (\text{restriction}) \quad (6.5)$$

that can be regarded as a linear map of the fusion algebras. Here the  $t$ 's are labels for the  $\mathcal{T}$ -particle spectrum and the  $n_a^t$ 's are multiplicities denoting the number of times  $t$  occurs in the restriction of  $a$ .

There are three important, physically motivated, conditions on these branching rules.

1. The condensed boson should restrict to the new vacuum (which we also denote by 0), so

$$b \rightarrow 0 + \sum_{t \neq 0} n_b^t t \quad (6.6)$$

2. The restriction should respect the fusion rules

$$\left( \sum_t n_a^t t \right) \times \left( \sum_t n_b^s s \right) = \sum_r N_{ab}^c n_c^r r \quad (6.7)$$

3. The restriction of particles and antiparticle should be consistent

$$\bar{a} \rightarrow \sum_t n_a^t \bar{t} \quad (6.8)$$

As a consequence of these assumptions and the uniqueness of the vacuum, 0 must restrict to the new vacuum

$$0 \rightarrow 0 \quad (6.9)$$

and the quantum dimensions are preserved

$$d_a = \sum_t n_a^t d_t \quad (6.10)$$

Using these conditions one can in general derive the particle spectrum and fusion rules of the  $\mathcal{T}$  theory.

Returning to the  $\mathfrak{su}(2)_4$  example, we see that  $4 \rightarrow 0$ , since  $d_4 = 1$  and therefore no more labels are allowed to occur in the restriction. Because  $4 \times 1 = 3$  and  $4 \times 3 = 1$ , we immediately conclude that 3 and 1 become indistinguishable after symmetry breaking. We will denote the corresponding  $\mathcal{T}$ -label by 1. Finally, let us look at the fusion of 2 with itself

$$2 \times 2 = 0 + 2 + 4 \rightarrow 0 + \sum_t n_2^t t + 0 \quad (6.11)$$

## 6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

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Since the vacuum appears twice on the right, the restriction of 2 must have more than one particle. Otherwise, this particle would be able to annihilate with itself in two different ways which is inconsistent with well defined fusion. Since  $d_2 = 2$ , this is possible and in fact leads us to conclude that  $2 \rightarrow 2_1 + 2_2$ , with  $d_{2_i} = 1$ . Hence, the broken phase has four sectors: 0, 1,  $2_1$  and  $2_2$ .

The fusion rules of 1 can be straightforwardly induced. Because in the unbroken phase we had  $1 \times 1 = 0 + 2$ , in the broken phase we have  $1 \times 1 = 0 + 2_1 + 2_2$ . This also shows that neither  $2_1$  nor  $2_2$  will be identical to the vacuum 0 in the broken phase, since this would imply the splitting of 1, which is impossible because  $d_1 < 2$ . Now, turning to the fusion rules for  $2_1$  and  $2_2$ , we find that they are either self-dual,  $2_1 \times 2_1 = 2_2 \times 2_2 = 0$ , or are dual to each other,  $2_1 \times 2_2 = 2_2 \times 2_1 = 0$ . Using the fusion rule  $2 \times 2 = 0 + 2 + 4$  and our current knowledge, we see that

$$\begin{aligned} 0 + 2_1 + 2_2 + 0 &= (2_1 + 2_2) \times (2_1 + 2_2) \\ &= 2_1 \times 2_1 + 2_1 \times 2_2 + 2_2 \times 2_1 + 2_2 \times 2_2 \end{aligned} \quad (6.12)$$

So if we assume that  $2_1$  and  $2_2$  are self-dual, we find that  $2_1 \times 2_2 = 2_1$  and  $2_2 \times 2_1 = 2_2$  or  $2_1 \times 2_2 = 2_2$  and  $2_2 \times 2_1 = 2_1$ . Either way, one easily concludes that associativity is violated by evaluating  $2_1 \times (2_2 \times 2_1)$  and  $(2_1 \times 2_2) \times 2_1$ . Thus, the fusion rules of the broken phase are symmetric and given by

$$\begin{aligned} 1 \times 1 &= 0 + 2_1 + 2_2 \\ 1 \times 2_1 &= 1 & 2_1 \times 2_1 &= 2_2 \\ 1 \times 2_2 &= 1 & 2_1 \times 2_2 &= 0 & 2_2 \times 2_2 &= 2_1 \end{aligned} \quad (6.13)$$

Note that the symmetry of the fusion rules is not a requirement in this formalism. Since  $\mathcal{T}$  does not have a consistent braiding, the fusion rules can be non-commutative.

### 6.1.2 Confinement

We now turn to the confinement projection, the second step of the formalism. In order to implement this step, a characterization of the sectors of  $\mathcal{T}$  that get confined is necessary.

In order to get a theory  $\mathcal{U}$  with well defined braiding, we must be able to assign twist factors to the charges of the  $\mathcal{U}$  theory. It is therefore logical to assume that, for an unconfined excitation  $u$  of the  $\mathcal{U}$ -theory, all charges of  $\mathcal{A}$  that restrict to  $u$  have the same twist factors.

Define the lift as the adjoint of the restriction.

$$t \rightarrow \sum_a n_a^t a, \quad (\text{lift}) \quad (6.14)$$

This lifts the charge  $t$  of  $\mathcal{T}$  into the theory  $\mathcal{A}$  as the sum of charges that have  $t$  in there restriction (including multiplicities) . The confinement condition can now be formulated as follows

## 6.1 Topological symmetry breaking

A charge  $t \in \mathcal{T}$  is *not confined* when  $\theta_a$  is the same for all  $a$  with  $n_a^t \neq 0$ . In that case, it belongs to  $\mathcal{U}$ .

It is sometimes useful to denote lifts of  $t$  as  $t^i$ . The condition above can then be reformulated as the statement that, for unconfined sectors  $u \in \mathcal{U}$ , one must have identical twist factors  $\theta_{u^i}$  for all lifts  $u^i$  of  $u$ .

Some physical requirements could be made separately on the set of unconfined particles. They should form a closed set under fusion, for instance, and they should of course contain the vacuum. The second requirement rules out condensates that do not have trivial-spin, which was to be expected. These extra requirements seem to be implied by the earlier condition, or are at least true in all worked out examples. We will come back to this when we discuss these kind of phase transitions again in relation to commutative algebra objects in braided tensor categories, exploiting the whole machinery that we have developed in previous chapters.

With the above definition of confinement in place, one may infer an interesting fact about the monodromy of the lifts of non-confined particles with the condensed excitations. Let  $u, v, w$  be three unconfined particles of the broken phase, and pick lifts that we denote as  $u^i, v^j, w^k$  with  $w^k \in u^i \times v^j$ . The monodromy of  $u^i$  and  $v^j$  in fusion channel  $w^k$  is by the ribbon equation governed by the combination of spin factors  $\theta_{w^k}/(\theta_{u^i}\theta_{v^j})$ . But since the twist factors of the lifts are the same as the twist factors of the lifted particle, we can write this as  $\theta_w/(\theta_u\theta_v)$ . When we apply this to  $u = 0, v = w$ , we find that the monodromy of unconfined particles with the lift of the vacuum is trivial. We could also take this latter statement as the definition. In that case, one can derive that the lifts of unconfined particles have identical spins, hence the two definitions are equivalent.

Now let us return to the  $\mathfrak{su}(2)_4$  example. Applying the confinement condition, we see that the restriction 1 of 1 and 3 gets confined. The bulk excitations are therefore the vacuum 0 and the ones we labelled  $2_1$  and  $2_2$ . These particle labels give the  $\mathcal{U}$ -theory in this case. It can be shown that the fusion rules for these particles admit precisely one solution to the pentagon and hexagon equations with these spin factors giving a theory known as  $\mathfrak{su}(3)_1$ . Usually, the labels are denoted as  $1, 3, \bar{3}$  instead of  $0, 2_1, 2_2$ .

$\mathfrak{su}(2)_4$ broken		
0	$d_0 = 1$	$h_0 = 0$
1	$d_1 = \sqrt{3}$	confined
$2_1$	$d_{2_1} = 1$	$h_{2_1} = \frac{1}{3}$
$2_2$	$d_{2_2} = 1$	$h_{2_2} = \frac{1}{3}$

**Figure 6.3:  $\mathfrak{su}(2)_4$  broken** - The quantum dimensions and spins for the broken phase of  $\mathfrak{su}(2)_4$ . The unconfined theory is also known as  $\mathfrak{su}(3)_1$ .

This concludes our recap of the topological symmetry breaking formalism. As a

## 6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

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last remark, let us point out interesting observations that were done regarding the total quantum dimension and the central charge of the theories involved. For the total quantum dimensions, the relation

$$\frac{\mathcal{D}_{\mathcal{A}}}{\mathcal{D}_{\mathcal{T}}} = \frac{\mathcal{D}_{\mathcal{T}}}{\mathcal{D}_{\mathcal{U}}} \quad (6.15)$$

was found. (This can actually be understood as a relation concerning the entropy of the different phases, as we will discuss in the next chapter). It was also found that the central charge is conserved under these kind of phase transition,

$$c_{\mathcal{A}} = c_{\mathcal{U}} \quad (6.16)$$

This is also nicely understood by the fact that in a CFT setting, topological symmetry breaking is dual to conformal extensions, as is discussed in [9].

The latter equation seems to limit the applicability of topological symmetry breaking to a restricted range of phase transitions. Phase transitions that change the value of  $c$  seem to lie outside the formalism. In [10], however, the formalism was applied in the context of the quantum Hall effect to multilayer systems. This way, a  $c$ -changing phase transition could be accounted for by incorporating an auxiliary layer, and the results were in agreement with the CFT approach in [40]. The multi-layer approach extends the applicability considerably.

### 6.2 Commutative algebra objects as Bose condensates

The study of the mathematical literature on commutative algebras in braided tensor categories, in relation to topological symmetry breaking was suggested because the formulas for the total quantum dimension central charge also appear in this context. In the next section, we will use this mathematics to extend the formulation of the topological symmetry breaking scheme and augment it with the diagrammatics developed in chapter 4. A pivotal aspect in this formulation is that that we identify the sectors  $t$  of  $\mathcal{T}$  with the lift  $\sum_t n_t^a$ . Hence diagrams for  $\mathcal{T}$  and  $\mathcal{U}$  can be interpreted as certain superpositions of diagrams of  $\mathcal{A}$ , allowing calculation using the  $F$ - and  $R$ -symbols from the  $\mathcal{A}$ -theory.

For this tensor category approach to topological symmetry breaking, we take a commutative algebra object in a modular tensor category [35, 48] as the definition of a bosonic condensate. Algebras in modular tensor categories have appeared in relation to physics, for example as boundary conditions in boundary CFT [35], but they have not really been related to symmetry breaking phase transitions and particle condensation.

We believe this is interesting for a couple of reasons. Tensor categories provide the appropriate language to discuss topological order in many cases. It is more concise and accurate than the explicit use of quantum groups and describes to the physical excitations. This becomes apparent, for example, in the formulation of the Levin-Wen models in terms of categories and the fact that it is not straight forward to treat the  $\mathfrak{su}(2)_k$  models as  $U_q[\mathfrak{su}(2)]$  representations. Also, the treatment of topological symmetry

breaking on the level of general quantum groups is likely to become cumbersome. The topological data captured by the  $F$ - and  $R$ -symbols, on the other hand, is physically intuitive.

Commutative algebra objects in modular tensor categories provide a rigorous way to construct the tensor category of the symmetry broken phase. Because all diagrams live in the  $\mathcal{A}$  theory, an extra benefit is that a sort of intermediate viewpoint can be attained. One can, so to say, look in between the transition from  $\mathcal{A}$  to  $\mathcal{T}$ . Related work (in progress) by Jesper Romers shows that this might be necessary when phase transitions in studying a lattice formulation of DGT numerically. Because diagrammatic calculations can be done, it is also possible to answer different questions. In the next chapter, we apply the formalism below to study the faith of the topological  $S$ -matrix under topologically symmetry breaking phase transitions.

We would like to point out that in a completely different context, there appeared evidence that condensates in the kind of systems under consideration carry an algebra structure (in the appropriate sense). In [16] the authors study the Kitaev model with boundary and show explicitly that introduction of the boundary leads to condensation of excitations that, collectively, are endowed with the structure of an algebra object in the category of representations of the underlying quantum double symmetry group.

For the discussion below, we made heavy use of [48].

### 6.2.1 The condensate

The intuition we should bear in mind for the tensor category approach to topological symmetry breaking is that charges of the intermediate theory  $\mathcal{T}$  are identified with their lift in  $\mathcal{A}$ . In particular, the new vacuum will be formed out of all particles that condense. This gives rise to a condensate

$$\phi = 0 + \delta_1 + \dots + \delta_n \tag{6.17}$$

We will use Greek indices for the condensed excitations from now on.<sup>1</sup> But for a satisfactory alternative description of the symmetry breaking scheme, we actually have to turn this logic around, since we do not want to rely an analysis along the lines of the previous section, or prior knowledge of the broken phase. In stead, we will have to guess the condensate (as we chose a boson to condense) and check that it obeys the right consistency conditions.

Therefore, fix some  $\phi = 0 + \delta_1 + \dots + \delta_n$ . Which consistency conditions should be required. We want to consider  $\phi$  as the new vacuum and the conditions on  $\phi$  can all be related back to this requirement.

As a first condition, in line with (A.13), if  $\delta \in \phi$  we should also have  $\bar{\delta} \in \phi$ . In other words, we require  $\bar{\phi} = \phi$ .

To keep the discussion transparent, we will often assume that there is only one non-trivial particle that condenses, so  $\phi = 0 + \delta$ . Note that in this situation,  $\bar{\phi} = \phi$  means

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<sup>1</sup>These should of course not be mistaken with the indices labelling vertices in chapter 4. The theory  $\mathcal{A}$  is assumed to be multiplicity free.

## 6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

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$\bar{\delta} = \delta$ . As a short hand for the charge line labelled by the condensate we introduce a striped line

$$\begin{array}{c} | \\ | \\ | \end{array} = \begin{array}{c} | \\ \uparrow \\ \phi \end{array} = \begin{array}{c} \dots \\ | \\ 0 \end{array} + \begin{array}{c} | \\ \uparrow \\ \delta \end{array} \quad (6.18)$$

where we can leave out orientation because  $\phi$  is self-conjugate.

With respect to charge lines labelled by a superposition of charges, which expand as a sum of diagrams, we point out that it is always possible to restrict to some subset of the charges, rendering all diagrams that end on other charges zero. This is accomplished by attaching specific charge lines on the outer ends. For example, connecting the vacuum line to one end of the condensate line kills the  $\delta$ . For clarity, we place a dot, since the vacuum line is invisible. So we have

$$\begin{array}{c} \bullet \\ | \\ | \\ | \end{array} = \left( \begin{array}{c} \dots \\ | \\ 0 \end{array} + 0 \cdot \begin{array}{c} | \\ \uparrow \\ \delta \end{array} \right) = \begin{array}{c} | \\ | \\ | \end{array} \quad (6.19)$$

Now we can conveniently express the consistency conditions for  $\phi$  using diagrammatic equations.

For  $\phi$  to be a well-defined condensate, we must be able to define a vertex

$$\begin{array}{c} \diagup \\ \diagdown \\ | \end{array} = \sum_{\alpha, \beta, \gamma \in \phi} M_{\gamma}^{\alpha\beta} \begin{array}{c} \alpha \\ \diagdown \\ \diagup \\ \gamma \\ | \end{array} \quad (6.20)$$

where the  $M_{\gamma}^{\alpha\beta}$  are complex coefficients, such that

1. 
$$\begin{array}{c} \bullet \\ \diagdown \\ \diagup \\ | \end{array} = \begin{array}{c} | \\ | \\ | \end{array} \quad (6.21)$$

2. 
$$\begin{array}{c} \diagup \\ \diagdown \\ | \end{array} = \begin{array}{c} \diagup \\ \diagdown \\ | \end{array} \quad (6.22)$$

3. 
$$\begin{array}{c} \diagup \\ \diagdown \\ | \end{array} = \begin{array}{c} \diagup \\ \diagdown \\ | \end{array} \quad (6.23)$$

Note, again, that we use Greek indices to label general charges in the condensate.

We can of course write these conditions purely in terms of the coefficients  $M_{\gamma}^{\alpha\beta}$  appearing in the condensate vertex. This would give

1. 
$$M_{\gamma}^{0\beta} = \delta_{\beta\gamma} \quad (6.24)$$

2. 
$$M_{\gamma}^{\beta\alpha} = R_{\gamma}^{\alpha\beta} M_{\gamma}^{\alpha\beta} \quad (6.25)$$

## 6.2 Commutative algebra objects as Bose condensates

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3.

$$\sum_{\mu \in \phi} M_{\mu}^{\alpha\beta} M_{\xi}^{\mu\gamma} [F_{\xi}^{\alpha\beta\gamma}]_{\mu\nu} = M_{\xi}^{\alpha\nu} M_{\nu}^{\beta\gamma} \quad (6.26)$$

These expressions are obtained by expanding the diagrams and comparing coefficients in front of basis vectors in the according splitting spaces, applying  $F$ -moves first when necessary. For example, the left hand side of the third consistency condition (6.23) expands in  $\mathcal{A}$  as

$$\sum_{\alpha, \beta, \gamma, \xi, \mu \in \phi} M_{\mu}^{\alpha\beta} M_{\xi}^{\mu\gamma} \begin{array}{c} \alpha \quad \beta \quad \gamma \\ \diagdown \quad \diagup \\ \mu \\ \diagup \\ \xi \end{array} = \sum_{\alpha, \beta, \gamma, \xi, \mu, \nu \in \phi} M_{\mu}^{\alpha\beta} M_{\xi}^{\mu\gamma} [F_{\xi}^{\alpha\beta\gamma}]_{\mu\nu} \begin{array}{c} \alpha \quad \beta \quad \gamma \\ \diagdown \quad \diagup \\ \nu \\ \diagup \\ \xi \end{array} \quad (6.27)$$

while the right hand side gives

$$\sum_{\alpha, \beta, \gamma, \xi, \nu \in \phi} M_{\xi}^{\alpha\nu} M_{\nu}^{\beta\gamma} \begin{array}{c} \alpha \quad \beta \quad \gamma \\ \diagdown \quad \diagup \\ \nu \\ \diagup \\ \xi \end{array} \quad (6.28)$$

Comparing the coefficients in front of the diagrams with corresponding charges leads to equation (6.26).

Before we discuss the physical meaning of these conditions, let us simply use them to simplify the expression for the vertex when  $\phi = 0 + \delta$ , i.e. in the case when there is only one condensed boson. Using (6.24) and (6.25) we see that the expression (6.20) is of the form

$$\begin{array}{c} \alpha \quad \beta \quad \gamma \\ \diagdown \quad \diagup \\ \mu \\ \diagup \\ \xi \end{array} = \begin{array}{c} 0 \quad 0 \quad 0 \\ \diagdown \quad \diagup \\ 0 \\ \diagup \\ 0 \end{array} + \begin{array}{c} 0 \quad \delta \quad \delta \\ \diagdown \quad \diagup \\ \delta \\ \diagup \\ \delta \end{array} + \begin{array}{c} \delta \quad 0 \quad 0 \\ \diagdown \quad \diagup \\ \delta \\ \diagup \\ \delta \end{array} + M_0^{\delta\delta\delta} \begin{array}{c} \delta \quad \delta \quad \delta \\ \diagdown \quad \diagup \\ 0 \\ \diagup \\ 0 \end{array} + M_{\delta}^{\delta\delta\delta} \begin{array}{c} \delta \quad \delta \quad \delta \\ \diagdown \quad \diagup \\ \delta \\ \diagup \\ \delta \end{array} \quad (6.29)$$

We have the freedom to put  $M_0^{\delta\delta} = 1$  as well. To see this, suppose we have found  $M_{\gamma}^{\alpha\beta}$  such that above conditions are all satisfied. Now choose arbitrary non-zero numbers  $c_{\alpha}$  for  $\alpha \in \phi$  with the restriction that  $c_0 = 1$ . Then define

$$\tilde{M}_{\gamma}^{\alpha\beta} = \frac{c_{\alpha} c_{\beta}}{c_{\gamma}} M_{\gamma}^{\alpha\beta} \quad (6.30)$$

One can readily see that, if the  $M_{\gamma}^{\alpha\beta}$  satisfy the conditions, so do the  $\tilde{M}_{\gamma}^{\alpha\beta}$ . Therefore we can switch to the latter just as well.<sup>1</sup> By choosing  $c_{\delta} = (M_0^{\delta\delta})^{-1/2}$  we get  $\tilde{M}_0^{\delta\delta} = 1$ .

So the one non-trivial coefficient that we should find is  $M \equiv M_{\delta}^{\delta\delta}$ . By using the third relation, we can express this in terms of the  $F$ -symbols by

$$M^2 = -\frac{\delta_{0c} - [F_{\delta}^{\delta\delta\delta}]_{0c}}{\delta_{\delta c} - [F_{\delta}^{\delta\delta\delta}]_{\delta c}}, \quad \forall c \in \mathcal{A} \quad (6.31)$$

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<sup>1</sup>Mathematically speaking, the  $c_{\alpha}$  define an isomorphism of (co)algebras.

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Here  $\delta_{xy}$  is the Kronecker  $\delta$ -function that tests if the arguments are equal, while the other  $\delta$ 's denote the condensed particle. This equation is most easily derived by attaching  $\delta$  on the outer lines of (6.23). Then this equation becomes

$$\begin{array}{c} \delta \swarrow \delta \searrow \\ \delta \uparrow \end{array} + M^2 \begin{array}{c} \delta \swarrow \delta \searrow \\ \delta \uparrow \end{array} = \begin{array}{c} \delta \swarrow \delta \searrow \\ \delta \uparrow \end{array} + M^2 \begin{array}{c} \delta \swarrow \delta \searrow \\ \delta \uparrow \end{array} \quad (6.32)$$

Applying the  $F$ -move on the left, and solving for  $M^2$  we find (6.31). This equation actually puts a strong constraint on the particle  $\delta$  since it must hold for all charges  $c$ . Note that this constraint was not present at all in the approach of the first section of this chapter.

Let us turn to the physical interpretation of the conditions on the condensate. Remember that we are looking to describe phase transitions induced by a bosonic condensate. The second condition, involving the braiding, is most clearly connected to this physical statement. It gives a more intricate condition on the particles in the condensate than just having trivial spin, but it makes sense that a stable condensate should have a state invariant under braiding. This condition therefore seems logical. For identical particles, it is also sensible to require invariance under braiding and not just under monodromy.

Let us take a closer look at the trivial braiding condition for condensate lines and what it implies for monodromy. By virtue of (6.22) we have

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ | \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ | \end{array} \quad (6.33)$$

Expanding both sides and using the monodromy equation this gives

$$\frac{\theta_\gamma}{\theta_\alpha \theta_\beta} = 1 \quad (6.34)$$

for all  $\alpha, \beta, \gamma \in \phi$  that give a vertex allowed by fusion. By taking  $\gamma = 0$ , we see that  $\theta_\alpha \theta_{\bar{\alpha}} = \theta_\alpha^2 = 1$  which leaves  $\theta_\alpha = \pm 1$  as the only possible solution for all  $\alpha \in \phi$ . Of course,  $\theta_\alpha = 1$ , is precisely what we expect from a bosonic condensate. Note that when we start with some  $\alpha$  with  $\theta_\alpha = 1$  which we require to condense, this condition generalizes the condition for a boson given before. Not only should there be a fusion channel  $\gamma \in \alpha \times \alpha$  with trivial spin, this fusion channel should be part of the condensate. The conditions make sure that for arbitrary particle number  $n$ , there is a fusion channel in  $\alpha^n$  that is invariant under monodromy (and even braiding). Hence, the particles in the condensate satisfy the general definition of a boson in [9] when  $\theta_\alpha = 1$ , *forall*  $\alpha \in \phi$ .

The exotic case that some  $\alpha \in \phi$  has  $\theta_\alpha = -1$  is truly rare. When  $\alpha \in \alpha \times \alpha$ , the monodromy equation gives  $\theta_\alpha / (\theta_\alpha \theta_\alpha) = 1$ , such that  $\theta_\alpha = -1$  is impossible. Furthermore, equation (6.22) implies that  $R_0^{\bar{\alpha}\alpha} = \theta_\alpha^* \mathbf{f}_\alpha^* = 1$ , so this can only happen

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for self-dual  $\alpha$  with Frobenius-Schur indicator  $\mathbf{fb}_\alpha$  equal to  $-1$ . We see that it possibly gives rise to fermionic condensates which might be interesting, but we have investigated this case yet. We will exclude this exotic case from the discussion for convenience.

The logic of the argument shows that when  $\theta_\alpha = 1$  for all  $\alpha$  in the condensate, the Frobenius-Schur indicators also equal unity. This means that conveniently, we do not have to keep track of the flags introduced earlier when working with the condensate.

The physical necessity of the other two consistency conditions is in a sense less clear, but have a clear interpretation when we agree that we want the condensate to be like the vacuum of a new theory. The third condition (6.23) says that we can freely reconnect the condensate lines; a sensible property for vacuum lines and in fact necessary when we look at the axioms of tensor categories. The first condition gives a rule to remove outer legs labelled by the condensate from diagrams by attaching a dot, which gives a way to pass from a tensor product state of two copies of the vacuum to a single copy, also a necessary condition for the vacuum to obey.

For concreteness, let us look again at the  $\mathfrak{su}(2)_4$  example. From the earlier discussion, we expect that the condensate  $\phi = 0 + 4$  should fit the new scheme. Since  $4 \notin 4 \times 4$ , the vertex simply becomes

$$\begin{array}{c} \diagup \\ \diagdown \\ | \end{array} = \begin{array}{c} 0 \quad 0 \\ \diagdown \quad \diagup \\ \vdots \\ 0 \end{array} + \begin{array}{c} 0 \quad 4 \\ \diagdown \quad \diagup \\ \uparrow \\ 4 \end{array} + \begin{array}{c} 4 \quad 0 \\ \diagdown \quad \diagup \\ \uparrow \\ 4 \end{array} + \begin{array}{c} 4 \quad 4 \\ \diagdown \quad \diagup \\ \vdots \\ 0 \end{array} \quad (6.35)$$

The Frobenius-Schur indicator for general  $\mathfrak{su}(2)_k$  theories is  $+1$  for particles with even labels and  $-1$  for particle with odd labels (corresponding to bosonic and fermionic representations for  $\mathfrak{su}(2)$  in the non- $q$ -deformed case). Hence,  $R_0^{44} = \mathbf{fb}_4\theta_4 = 1$  and (6.22) is satisfied. Condition (6.21) and (6.21) are also trivially satisfied in this case. So the condensate  $\phi = 0 + 4$  indeed fits the scheme.

An interesting observation is that the rules for the condensate allow us to construct a family of states for arbitrary particle number with remarkable invariance properties. In fact, if we allow superposition of anyonic charges, we can conveniently write down a state formed by stacking condensate vertices together. For example

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \vdots \end{array} \quad (6.36)$$

This expands in  $\mathcal{A}$  as a sum of states with mixed particle number. By connecting the outer legs to lines labelled by single charges from the condensate we can project to a state with fixed particle number and fixed charge on the outer legs. By using equations (6.22) and (6.23) we can show that is invariant under the full braid operator

$$\begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \vdots \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \vdots \\ | \end{array} \quad (6.37)$$

This is a hint that we indeed have the right definition of bosonin condensate. Note that the most general conditions discussed in [9] are satisfied if this definition holds in

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a particularly elegant way, and we have a constructive way to build states that are fully invariant under braiding and not just monodromy.

The invariance properties of states of the form (6.36) suggest that they might be connected to the ground state of the broken phase. In lattice models with a finite size lattice, this state can be concretely realized by operators acting on the vacuum. In the Kitaev model with a boundary for example, an important role is played by so called ribbon operators and (6.36) can essentially be interpreted as an expression in terms of these operators. It would be interesting to investigate the ground state of the symmetry broken phase in these kind of lattice models and see if they relate to the diagram in (6.36). Examples could be the Kitaev model with boundary [17] where the boundary conditions implement an explicit symmetry breaking mechanism, or lattice formulations of discrete gauge theories as in for example [13].

In any case, the conditions 1–3 have interesting physical consequences and interpretations. Let us proceed with the construction of a consistent tensor category based on these. The next step is the construction of the particle spectrum of the  $\mathcal{T}$ -theory.

### 6.2.2 The particle spectrum of $\mathcal{T}$

Now that we have defined the vacuum of the broken theory, i.e. the condensate  $\phi$ , it is time to introduce particles. In order to create a diagrammatic formalism, we will not only need to know the particle labels that occur in the sum, but also how to construct diagrams. A minimal requirement is therefore, that we define how the particles couple to the vacuum. This leads to the definition of a vertex obeying certain consistency conditions, much like the vertex for the condensate.

Charges of  $\mathcal{T}$  will be denoted with labels  $t, s, r, \dots$ , and we write

$$\left| \begin{array}{c} \uparrow \\ t \end{array} \right. = \sum_{a \in t} \left| \begin{array}{c} \uparrow \\ a \end{array} \right. = \sum_a n_a^t \left| \begin{array}{c} \uparrow \\ a \end{array} \right. \quad (6.38)$$

Now, how do we find what  $a \in t$ , i.e. which particles  $a$  restrict to  $t$ , if we do not want to repeat the analysis along the lines of the previous section? To give an idea, we will momentarily switch back to the notation of the previous section. Recall the symmetries of the fusion multiplicities

$$N_{ab}^c = N_{ba}^c = N_{bc}^{\bar{a}} = N_{\bar{a}c}^{\bar{b}} \quad (6.39)$$

Now note that condition 2 and 3 on the branching rules, i.e. equation (A.12) and (3), can be summarized by the equalities

$$\sum_{t,s} n_a^t n_b^s N_{ts}^r = \sum_c N_{ab}^c n_c^r \quad (6.40)$$

and

$$n_{\bar{a}}^{\bar{t}} = n_a^t \quad (6.41)$$

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Using these, we find

$$\sum_b N_{ab}^c n_b^0 = \sum_{t,s} n_a^t n_c^s N_{t0}^s \quad (6.42)$$

$$= \sum_t n_a^t n_c^t \quad (6.43)$$

Writing the condensate as the lift of the vacuum,  $\phi = \sum_b n_b^0$ , this gives

$$\phi \times a = \sum_{b,c} n_b^0 N_{ba}^c c = \sum_t n_a^t n_c^t c \quad (6.44)$$

In words, we see that  $\phi$  falls apart as the sum of lifts of the charges  $t$  in the restriction of  $a$ . Identifying  $t$  with its lift, we might say that we find all  $t$  in the restriction of  $a$  as parts of  $\phi \times a$ . In fact, a lemma 3.4 in [48] assures that the particles of the broken phase can all be found this way, in the tensor category approach to topological symmetry breaking.

In the  $\mathfrak{su}(2)_4$  example, we indeed find back the particle spectrum of  $\mathcal{T}$  in a particularly quick way. With  $\phi = 0 + 4$  the fusion rules (6.3) give

$$\phi \times 0 = 0 + 4$$

$$\phi \times 1 = 1 + 3$$

$$\phi \times 2 = 2 + 2$$

$$\phi \times 3 = 1 + 3$$

$$\phi \times 4 = 0 + 4$$

This shows 4 gets identified with the vacuum, 1 and 3 get identified with each other in the broken phase, in agreement with the earlier discussion. It also shows that the particle 2 will split into two components as we found before.

One more important observation we can make, is that the multiplicity of  $a$  in the lift of  $t$  is equal to the multiplicity of  $t$  in the restriction of  $a$ , since they are both given by  $n_a^t$ .<sup>1</sup> Multiplicities greater  $n_a^t > 1$  do not seem to appear, although they might in exotic examples. To keep the discussion streamlined, we will assume that they do not occur, such that  $a \in t$  occurs at most once. This implies that splitting can be seen directly from  $\phi \times a$  as the occurrence of multiple copies of  $a$ . For  $\phi = 0 + \delta$ , we see that  $a$  splits iff  $a \in \delta \times a$ .

Now, we will look at the coupling of the  $\mathcal{T}$  particles with the condensate, for which we need to define the vertex

$$\begin{array}{c} \diagup \\ \diagdown \\ \uparrow t \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \uparrow t \end{array} = \sum_{\substack{\alpha \in \phi \\ a, b \in t}} {}^{(t)}A_b^{\alpha a} \begin{array}{c} \diagdown \\ \diagup \\ \uparrow b \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \uparrow a \end{array} \quad (6.45)$$

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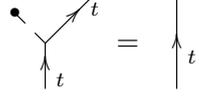
<sup>1</sup>In mathematical language: The corresponding functors are each others adjoint (left and right). This is morally the same as Frobenius reciprocity for group representations.

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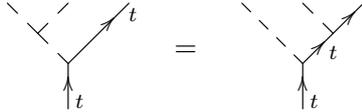
The coefficients  ${}^{(\theta)}A_b^{\alpha a}$  must be such that some consistency conditions are satisfied, namely

1.



$$(6.46)$$

2.



$$(6.47)$$

This gives the following equations

1.

$${}^{(\theta)}A_b^{0a} = \delta_{ab}, \quad \forall a, b \in s \quad (6.48)$$

2.

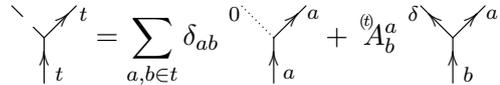
$$\sum_c M_\gamma^{\alpha\beta} {}^{(\theta)}A_b^\gamma [F_b^{\alpha\beta a}]_{\gamma c} = {}^{(\theta)}A_c^{\beta a} {}^{(\theta)}A_b^{\alpha c} \quad (6.49)$$

Formally, we might view a charge  $t$  as the combination of a sum of fields of  $\mathcal{A}$  together with the set of coefficients

$$t = \left( a_1 + \cdots + a_n, {}^{(\theta)}A_{a_j}^{\alpha a_i} \right) \quad (6.50)$$

Indeed, if either one is different for some  $s$  and  $t$ , we must regard  $s$  and  $t$  to be different particles of  $\mathcal{T}$ . This means that some particles of  $\mathcal{T}$  can consist of the same sum of  $\mathcal{A}$ -charge but only differ in the coefficients  ${}^{(\theta)}A_b^{\alpha a} / {}^{(\theta)}A_{b'}^{\alpha' a'}$ , corresponding to different  $\mathcal{T}$ -sectors with identical lifts. This raises a subtle issue. Because there is a certain freedom in the choice of  ${}^{(\theta)}A_b^{\alpha a}$  and a notion of isomorphism of charges of  $\mathcal{T}$ , when do we know that  $\mathcal{T}$  charges are different or the same. We will come back to this.

Let us work out the case  $\phi = 0 + \delta$ . The first conditions (6.46) says that only the  ${}^{(\theta)}A_b^{\delta a}$  can be non-trivial, so we leave out the index  $\delta$  in this case. The vertex then becomes



$$(6.51)$$

The second condition gives

$$\delta_{ab} [F_a^{\delta\delta a}]_{0c} + M {}^{(\theta)}A_b^a [F_b^{\delta\delta a}]_{\delta c} = {}^{(\theta)}A_c^a {}^{(\theta)}A_b^c \quad (6.52)$$

From here, we can deduce some useful expressions to facilitate the calculation of the  ${}^{(\theta)}A_a^a$  in concrete examples.

- By choosing  $a \neq b$  and  $a = c$  we find

$${}^{(\theta)}A_a^a = M [F_b^{\delta\delta a}]_{\delta a}, \quad a \neq b, a, b \in t \quad (6.53)$$



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There are indeed two solutions  ${}^{(2)}A_2^2 = \pm 1$ , and we will denote the corresponding particles by  $2_{\pm}$ . So  ${}^{(2+)}A_2^2 = +1$  and  ${}^{(2-)}A_2^2 = -1$ . The corresponding vertices are thus

$$\begin{array}{c} \diagup \\ \diagdown \\ \uparrow \\ 2_+ \end{array} = \begin{array}{c} 0 \\ \diagup \\ \diagdown \\ \uparrow \\ 2 \end{array} + \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ \uparrow \\ 2 \end{array} \quad (6.60)$$

and

$$\begin{array}{c} \diagup \\ \diagdown \\ \uparrow \\ 2_+ \end{array} = \begin{array}{c} 0 \\ \diagup \\ \diagdown \\ \uparrow \\ 2 \end{array} - \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ \uparrow \\ 2 \end{array} \quad (6.61)$$

It is now easy to verify the conditions (6.47) and (6.47) when we plug in these definitions of the vertices using the  $F$ -moves for the  $\mathfrak{su}(2)_4$ -theory.

As occurred above, some combinations of charges do not occur for the  ${}^{(t)}A_b^{\alpha a}$ , namely when  $N_{\alpha a}^b = 0$  and the vertex is not allowed by fusion. In this case we put  ${}^{(t)}A_b^{\alpha a} = 0$  by convention. Just as well, we can take  ${}^{(t)}A_b^{\alpha a} = 0$  when  $a, b \notin t$ . However, when  $\alpha \in \phi$ ,  $a, b \in t$  and the vertex is allowed by fusion,  ${}^{(t)}A_b^{\alpha a}$  will *always be non-zero* for a particle  $t$ . This is a kind of irreducibility condition, distinguishing true particles  $t$  from a superposition of particles  $t_1 + \dots + t_n$ , for which solutions to the consistency conditions may exist for which zero coefficients do occur. These basically correspond to the projection onto a subset of the particles that occur in the superposition.

Above, we mentioned that the particles of the broken phase can all be found in the fusion products  $\phi \times a$ . Let us give a graphical illustration of why the defining conditions have to hold for  $\phi \times a$ , hence either this corresponds to a particle of the broken theory or is a superposition of particles of the broken theory. The charge line labelled  $\phi \times a$  can also be written as two parallel lines, one labelled  $\phi$  the other  $a$ . Hence we have

$$\begin{array}{c} \uparrow \\ \phi \times a \end{array} = \begin{array}{c} | \\ | \\ \uparrow \\ a \end{array} \quad (6.62)$$

We can use the condensate vertex to define a vertex for  $\phi \times a$ .

$$\begin{array}{c} \diagup \\ \diagdown \\ \uparrow \\ \phi \times a \end{array} \equiv \begin{array}{c} \diagup \\ \diagdown \\ | \\ a \end{array} \quad (6.63)$$

The conditions on the condensate now give the requirements stated above.

$$\begin{array}{c} \bullet \\ \diagup \\ \diagdown \\ | \\ a \end{array} = \begin{array}{c} | \\ | \\ \uparrow \\ a \end{array} \quad \text{and} \quad \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \\ | \\ a \end{array} = \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \\ | \\ a \end{array} \quad (6.64)$$

At this point we have constructed a description of all particles of the new theory, which includes the definition of certain vertices involving the new vacuum. Note that we have not derived the fusion rules, nor did we need to. They can be inferred by the methods of section 6.1, but it would be nice to derive them in a more straight forward manner in the present context or even have a closed formula for the  $N_{ts}^r$  in terms of the  $N_{ab}^c$ . The issue is subtle, because the definition of the tensor product in the category  $\mathcal{T}$  is non-trivial. This deserves more attention, but we have no results on the subject yet.

### 6.2.3 Derived vertices

The definition of the vertices as in (6.20) and (6.45) allow us to derive several vertices. The condensate vertex with two legs down is defined as

$$\begin{array}{c} | \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \sum_{\alpha, \beta, \gamma \in \phi} (M_{\gamma}^{\beta\alpha}) \begin{array}{c} \uparrow \gamma \\ \diagdown \quad \diagup \\ \alpha \quad \beta \end{array} \quad (6.65)$$

We furthermore define

$$\begin{array}{c} \uparrow s \\ \diagdown \quad \diagup \\ s \quad \diagdown \end{array} = \sum_{\substack{\alpha \in \phi \\ a, b \in s}} ({}^s A_b^a)^* \begin{array}{c} \uparrow b \\ \diagdown \quad \diagup \\ a \quad \alpha \end{array} \quad (6.66)$$

This is convenient for the discussion of the  $S$ -matrix for the broken phase. Formally, these morphisms correspond to the duals of the splitting versions.

With these definitions in place, we find

$$\begin{array}{c} | \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ | \end{array} = q \begin{array}{c} | \\ | \\ | \\ | \end{array} \quad (6.67)$$

with  $q = d_{\phi}$ , at least in all our examples.

### 6.2.4 Confinement

The confinement in the present context is the same as it was in the earlier discussion. Recall that unconfined particles were characterized by the condition that all charges in the lift have identical twist factors, or equivalently, that the lift has trivial monodromy with the lift of the vacuum, i.e. the condensate. The second condition, which is in fact slightly more general as it can be used for condensates with  $\theta_{\alpha} \neq 1$ , is nicely expressed diagrammatically as

$$\begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \uparrow t \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \uparrow t \end{array} \quad (6.68)$$

Using the monodromy equation, we find that this is equivalent to

$$\theta_a / (\theta_{\alpha} \theta_b) = 1, \quad \forall a, b \in t, \alpha \in \phi \quad (6.69)$$

Because  $\theta_{\alpha} = 1$  for all particles  $\alpha$  of the condensate, we indeed recover the condition that

$$\theta_a = \theta_b, \quad \forall a, b \in t \quad (6.70)$$

which is precisely the statement that all lifts have equal topological spin.



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The fact that the projector evaluates to zero on confined particles  $t$  is a bit subtle. Imagine that we have some operator  $X$ , which has the property that for particles  $t \in \mathcal{T}$  we have

$$\begin{array}{c} \diagup \\ \downarrow \\ \boxed{X} \\ \uparrow \\ \diagdown \end{array} \begin{array}{c} \uparrow \\ t \end{array} = \begin{array}{c} \diagup \\ \downarrow \\ \boxed{X} \\ \uparrow \\ \diagdown \end{array} \begin{array}{c} \uparrow \\ t \end{array} \quad (6.74)$$

Since  $X$  is a single anyon operator, we know that on anyons of  $\mathcal{A}$ , it acts by multiplication by a complex number

$$\begin{array}{c} \uparrow \\ \boxed{X} \\ \uparrow \end{array} = \sum_a X_a \begin{array}{c} | \\ a \end{array} \quad (6.75)$$

Now, by expanding both sides of (6.74) as diagrams in  $\mathcal{A}$ , we find that

$${}^{(t)}A_b^{\alpha a} X_a = {}^{(t)}A_b^{\alpha a} X_b, \quad \forall a, b \in t \quad (6.76)$$

This implies  $X_a = X_b$  for all  $a, b \in s$ , unless  ${}^{(t)}A_b^{\alpha a} = 0$  for all  $\alpha \in \phi$  for some  $a$  and  $b$ . The latter case can actually not occur, since this would mean that  $a$  and  $b$  do not occur in the same particle  $t$ . Or in other words, what we thought was a particle  $t$  is actually not a single particle of  $\mathcal{T}$  but splits as a sum of particles. This is analogous to Schur's lemma.

The operator  $P_t$  has the property expressed under (6.74), hence it acts by multiplication on the charge line labelled  $t$ . Say,

$$\begin{array}{c} \uparrow \\ \boxed{P_t} \\ \uparrow \end{array} = c_t \begin{array}{c} | \\ t \end{array} \quad (6.77)$$

Now consider a situation, where we apply the projector  $P_t$  on a  $t$  in the broken phase, followed by a counter clockwise  $2\pi$  rotation of the system. The corresponding operator is

$$\begin{array}{c} \circlearrowleft \\ \diagup \\ \downarrow \\ \boxed{P_t} \\ \uparrow \\ \diagdown \end{array} \begin{array}{c} \uparrow \\ t \end{array} = \begin{array}{c} \diagup \\ \downarrow \\ \boxed{P_t} \\ \uparrow \\ \diagdown \end{array} \begin{array}{c} \uparrow \\ t \end{array} \quad (6.78)$$

where the equality is shown by applying (6.47) and the monodromy equation, and using that the twist of  $\phi$  is trivial. From the presentation on the right of equation (6.78) we see that this operator obeys (6.74). Hence, it also acts by multiplication by a complex

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number,

$$\text{Diagram} = c'_t \text{Diagram} \quad (6.79)$$

But on the other hand, we can calculate

$$\text{Diagram} = \sum_{a \in t} c_t \theta_a \text{Diagram} \quad (6.80)$$

If  $c_t \neq 0$  then this shows that we must have  $\theta_a = c_t/c'_t$  for any  $a \in t$ , which contradicts the confinement condition when  $t$  is confined.

It is an instructive exercise to calculate the diagram for  $P_t$  explicitly, when  $\phi = 0 + \delta$ . Then one has

$$\text{Diagram} = \sum_{a \in t} \left( \text{Diagram}_1 + \delta \text{Diagram}_2 + M^{(\theta)} A_a^a \delta \text{Diagram}_3 \right) \quad (6.81)$$

Using the monodromy equation and a sequence of  $F$ -moves, a diagrammatic calculation gives

$$\text{Diagram} = \sum_c \frac{\theta_c}{\theta_\delta \theta_a} [F_a^{\delta \delta a}]_{\delta c} [F_a^{\delta \delta a}]_{0c}^* d_\delta \text{Diagram} \quad (6.82)$$

Thus we find

$$\frac{1}{q} \text{Diagram} = \frac{1}{q} \sum_{a \in t} \left( 1 + \frac{S_{\delta a}}{S_{0a}} + M^{(\theta)} A_a^a \left[ \sum_c \frac{\theta_c}{\theta_\delta \theta_a} [F_a^{\delta \delta a}]_{\delta c} [F_a^{\delta \delta a}]_{0c}^* d_\delta \right] \right) \text{Diagram} \quad (6.83)$$

From this expression, it is not evident that the coefficient is either one or zero. But plugging in the data of the  $\mathfrak{su}(2)_4$  or  $\mathfrak{su}(2)_{10}$  example indeed gives zero for the coefficient in front of the charge  $a \in t$  when  $t$  is confined, and one if it is not.

This projector implies an interesting equality. Taking the quantum trace in (6.78), a diagrammatic manipulation shows that this gives

$$= q \sum_{a \in t} \theta_a d_a \quad (6.83)$$

while on the other hand, we showed that it is zero when  $t$  is confined. Hence we obtain

$$\sum_{a \in t} \theta_a d_a = 0 \quad \text{when } t \text{ is confined} \quad (6.84)$$

### 6.3 Breaking $\mathfrak{su}(2)_{10}$

The  $\mathfrak{su}(2)_{10}$  theory is an interesting example to consider. It has a bosonic charge that can condense with more exceptional properties than the  $\mathfrak{su}(2)_4$  theory.<sup>1</sup> The quantum dimensions and topological spins of the charges of  $\mathfrak{su}(2)_{10}$  are

$\mathfrak{su}(2)_{10}$		
0	$d_0 = 1$	$h_0 = 0$
1	$d_1 = \sqrt{s + \sqrt{3}}$	$h_1 = \frac{1}{16}$
2	$d_2 = 1 + \sqrt{3}$	$h_2 = \frac{1}{6}$
3	$d_3 = \sqrt{2} + \sqrt{2 + \sqrt{3}}$	$h_3 = \frac{5}{16}$
4	$d_4 = 2 + \sqrt{3}$	$h_4 = \frac{1}{2}$
5	$d_5 = 2\sqrt{2 + \sqrt{3}}$	$h_5 = \frac{35}{48}$
6	$d_6 = 2 + \sqrt{3}$	$h_6 = 1$
7	$d_7 = \sqrt{2} + \sqrt{2 + \sqrt{3}}$	$h_7 = \frac{21}{16}$
8	$d_8 = 1 + \sqrt{3}$	$h_8 = \frac{5}{3}$
9	$d_9 = \sqrt{2 + \sqrt{3}}$	$h_9 = \frac{33}{16}$
10	$d_{10} = 1$	$h_{10} = \frac{5}{2}$

The particle label 6 has trivial twist so it makes sense to try the condensate  $\phi = 0 + 6$ .

<sup>1</sup>The term ‘exceptional’ gets a precise meaning in the quantum version of the McKay correspondence. The condensates in the  $\mathfrak{su}(2)_k$  theories can be classified by Dynkin diagrams, in a fashion similar to the classical McKay correspondence. See [48].

## 6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

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Fusion following the general rule (4.1) gives

$$\begin{aligned}
\phi \times 0 &= 0 + 6 &&= (0 + 6) \\
\phi \times 1 &= 1 + 5 + 7 &&= (1 + 5 + 7) \\
\phi \times 2 &= 2 + 4 + 6 + 8 &&= (2 + 4 + 6 + 8) \\
\phi \times 3 &= 3 + 3 + 5 + 7 + 9 &&= (3 + 5 + 9) + (3 + 7) \\
\phi \times 4 &= 2 + 4 + 4 + 6 + 8 + 10 &&= (2 + 4 + 6 + 8) + (4 + 10) \\
\phi \times 5 &= 1 + 3 + 5 + 5 + 7 + 9 &&= (1 + 5 + 7) + (3 + 5 + 9) \\
\phi \times 6 &= 0 + 2 + 4 + 6 + 6 + 8 &&= (0 + 6) + (2 + 4 + 6 + 8) \\
\phi \times 7 &= 1 + 3 + 5 + 7 + 7 &&= (1 + 5 + 7) + (3 + 7) \\
\phi \times 8 &= 2 + 4 + 6 + 8 &&= (2 + 4 + 6 + 8) \\
\phi \times 9 &= 3 + 5 + 9 &&= (3 + 5 + 9) \\
\phi \times 10 &= 4 + 10 &&= (4 + 10)
\end{aligned}$$

On the outer right hand side we grouped the labels in the unique way, such that they nicely split up all the fusion products. These are the sums corresponding to the particles of  $\mathcal{T}$ . So, for example, we see that 8 and 2 get identified in the broken phase. The fields that split are 3, 4, 5, 6 and 7. An interesting feature is that the condensed particle 6 splits.

The equation for  $M$  is

$$M^2 = -\frac{\delta_{0c} - [F_6^{666}]_{0c}}{\delta_{\delta c} - [F_6^{666}]_{6c}} = -\sqrt{2} \quad (6.85)$$

which is indeed true for all charges  $c$ . We choose one root and define the condensate vertex as

$$\begin{array}{c} \diagup \\ \diagdown \\ | \end{array} = \begin{array}{c} 0 \\ \diagdown \\ \vdots \\ 0 \end{array} + \begin{array}{c} 0 \\ \diagdown \\ \uparrow 6 \end{array} + \begin{array}{c} 6 \\ \diagdown \\ \uparrow 6 \end{array} + \begin{array}{c} 6 \\ \diagdown \\ \vdots \\ 0 \end{array} + 2^{1/4}i \begin{array}{c} 6 \\ \diagdown \\ \uparrow 6 \end{array} \quad (6.86)$$

Using the  $F$ -symbols, one can check that equation (6.23) is satisfied.

We will not try to obtain all the  ${}^{\theta}A_b^a$  for this theory. But in later calculations we will need the  ${}^{\theta}A_a^a$ . These are most easily obtained by equation (6.53). To calculate  ${}^{\theta}A_a^a$ , we just choose a  $b \in t$  that is not equal to  $a$ , something that is always possible in this case, and apply

$${}^{\theta}A_a^a = 2^{1/4}i [F_b^{66a}]_{6a} \quad (6.87)$$

This way we obtain the following table

$t$	$A_a^t$	
(1 + 5 + 7)	$A_1^1 = 0$ $A_7^1 = 2^{1/4}i \cdot \sqrt{2 - \sqrt{3}}$	$A_5^1 = 2^{1/4}i \cdot (1 + \sqrt{3})^{-1/2}$
(2 + 4 + 6 + 8)	$A_2^2 = 0$ $A_6^2 = 2^{1/4}i \cdot \frac{1}{2}(1 - \sqrt{3})$	$A_4^2 = 2^{1/4}i \cdot \frac{1}{2}(-1 + \sqrt{3})$ $A_8^2 = 0$
(3 + 5 + 9)	$A_3^3 = -2^{1/4}i \cdot \sqrt{2 - \sqrt{3}}$ $A_9^3 = 0$	$A_5^3 = -2^{1/4}i \cdot (1 + \sqrt{3})^{-1/2}$
(3 + 7)	$A_3^3 = 2^{1/4}i \cdot \frac{1}{2}\sqrt{2}$	$A_7^3 = -2^{1/4}i \cdot \frac{1}{2}\sqrt{2}$
(4 + 10)	$A_4^4 = -2^{1/4}i$	$A_{10}^4 = 0$

(6.88)

Let us illustrate the usefulness by calculating the evaluation of the confinement projector, as stated in (6.83). Denote

$$C_a \equiv \sum_c \frac{\theta_c}{\theta_a} [F_a^{66a}]_{6c} [F_a^{66a}]_{10c}^* d_6 \quad (6.89)$$

One has

$a$	0	1	2	3	4	5	6	7	8	9	10
$C_a$	0	0	0	$-1 - \sqrt{3}$	$\sqrt{6}$	0	$-\sqrt{6}$	$1 + \sqrt{3}$	0	0	0

(6.90)

We also have

$a$	0	1	2	3	4	5	6	7	8	9	10
$\frac{S_{6a}}{S_{0a}}$	$2 + \sqrt{3}$	-1	-1	1	$2 - \sqrt{3}$	-1	$2 - \sqrt{3}$	1	-1	-1	$2 + \sqrt{3}$

(6.91)

Now we can calculate the coefficient in (6.83)

As an example, we work out the case  $t = (3 + 7)$  and  $t = (1 + 5 + 7)$ .

The excitation  $t = (3 + 7)$  of  $\mathcal{T}$  is not confined. We have

$$1 + \frac{S_{63}}{S_{03}} + M A_3^3 C_3 = 1 + 1 + \sqrt{2} \frac{1}{2} \sqrt{2} (1 + \sqrt{3}) = 3 + \sqrt{3} = q \quad (6.92)$$

since  $q = d_0 + d_6 = 1 + 2 + \sqrt{3} = 3 + \sqrt{3}$ . The same holds when we take  $a = 7$ . Dividing out  $q$  indeed gives 1, as claimed.

Now take  $t = (1 + 5 + 7)$ , and calculate the coefficient for  $a = 7$ . This gives

$$1 + \frac{S_{67}}{S_{07}} + M A_7^7 C_7 = 1 + 1 - \sqrt{2} \sqrt{2 - \sqrt{3}} (1 + \sqrt{3}) = 0 \quad (6.93)$$

in accordance with the fact that  $t$  is confined.

## 6. BOSE CONDENSATION IN TOPOLOGICALLY ORDERED PHASES

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## CHAPTER 7

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### Indicators for topological order

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Because topologically ordered phases defy the Landau symmetry breaking scheme of characterization by local order parameters, other indicators not based on symmetry have appeared in the literature. These are ground state degeneracy on surfaces of non-zero genus [32, 77], fractional of quantum numbers (charge [55], spin [2]), gapless edge excitations [78], and topological entanglement entropy [58, 66]. It is clearly of interest to know the behaviour of these indicators under a topological symmetry breaking phase transition. The topological entanglement entropy, that has been related to the total quantum dimension, shows an interesting relation with the quantum dimension of the condensate. This condensate quantum dimension  $q = d_\phi$  seems an interesting number characterizing the phase transition. It can be computed as the ratio between the quantum dimensions of the lift of an arbitrary sector  $t$  of the broken phase and the quantum dimension of  $t$  itself. One can show that the increase in topological entanglement entropy in each step of the phase transition is  $\log \sqrt{q}$  for a disc-like geometry.

Another interesting problem is to construct ‘topological order parameters’, operators that can be used to ‘measure’ the kind of topological order and signal phase transitions, in simulations or experiments. The topological  $S$ -matrix is a particularly interesting object. Through interferometry measurements it might become possible soon to obtain matrix elements of the topological  $S$ -matrix in, for example, FQH states. On the other hand, it decodes a lot of information about the underlying theory. In this chapter, we will also show how to obtain the topological  $S$ -matrix using the tensor category approach to topological symmetry breaking.

## 7.1 Topological entanglement entropy

For a quantum system described by a density matrix  $\rho$ , the von Neumann entropy  $S_E$  is defined as

$$S_E = -\text{Tr } \rho \log \rho \quad (7.1)$$

For a gapped system in a planar region with boundary of length  $L$ , it takes the form

$$S_E = \alpha L - \gamma + \dots \quad (7.2)$$

The ellipses represent terms that vanish in the limit  $L \rightarrow \infty$ . The universal additive constant  $-\gamma$  characterizes global features of the entanglement in the ground state and has been called the *topological entanglement entropy*. It signals non-trivial topological order and in [58, 66] it was shown that for a disc-like geometry it relates to the total quantum dimension of the theory underlying the topological order as

$$\gamma = \log \mathcal{D} \quad (7.3)$$

In a topological symmetry breaking phase transition  $\mathcal{A} \rightarrow \mathcal{T} \rightarrow \mathcal{U}$ , the relation between the total quantum dimensions  $\mathcal{D}_{\mathcal{A}}$ ,  $\mathcal{D}_{\mathcal{T}}$  and  $\mathcal{D}_{\mathcal{U}}$  therefore relates the topological entropy of the different phases. In particular the topological entanglement entropy increases by an amount

$$\gamma_{\mathcal{A}} - \gamma_{\mathcal{U}} = \log \frac{\mathcal{D}_{\mathcal{A}}}{\mathcal{D}_{\mathcal{U}}} \quad (7.4)$$

We will show that this ratio is directly related to  $q = d_\phi$ , the quantum dimension of the condensate. This is in fact equal to

$$q = \frac{\sum_a n_a^t a}{d_t} \quad (7.5)$$

for any  $t$  in the  $\mathcal{T}$ -theory.

We will prove that the right hand side of equation (7.5) is indeed not depending on the choice of  $t$ . But let us first make a remark on notational conventions and assumptions for this section. Throughout this section, we use the notation and assumptions and terminology of the original approach to topological quantum group symmetry breaking. In the literature on commutative algebras in tensor categories the same result is present (see [48]). But it is interesting that one can derive this from less involved assumptions.

Now, to show (7.5), we will just need the description of the theories on the level of the fusion algebras which we will also denote with  $\mathcal{A}$ ,  $\mathcal{T}$  and  $\mathcal{U}$ . The fusion product is written  $|a\rangle \times |b\rangle = \sum_c N_{ab}^c |c\rangle$  for  $\mathcal{A}$ . This defines the fusion matrices  $N_a$  by  $(N_a)_b^c = N_{ab}^c$ . For  $\mathcal{T}$  we use the same notations, but with  $\{a, b, c\} \rightarrow \{t, r, s\}$ . Recall that the vector

$$|\omega_{\mathcal{A}}\rangle = \sum_a d_a |a\rangle \quad (7.6)$$

is a common Perron-Frobenius eigenvector for the fusion matrices  $N_a$  with eigenvalue equations

$$N_a |\omega_{\mathcal{A}}\rangle = d_a |\omega_{\mathcal{A}}\rangle \quad (7.7)$$

where  $d_a$  is the quantum dimension of the charge  $a$ . This can be checked explicitly from the relation

$$d_a d_b = N_{ab}^c d_c \quad (7.8)$$

Similarly,

$$|\omega_{\mathcal{T}}\rangle = \sum_t d_t |t\rangle \quad (7.9)$$

is a common Perron-Frobenius eigenvector for the fusion matrices  $N_t$ , even though these do not necessarily commute.

The phase transition is described by the restriction map, as discussed in the previous chapter. This is conveniently written as  $B: \mathcal{A} \rightarrow \mathcal{T}$ . The lift is regarded as the transpose map  $B^\dagger: \mathcal{T} \rightarrow \mathcal{A}$ . Written out explicitly, we have

$$B|a\rangle = \sum_t n_a^t |t\rangle \quad , B^\dagger|t\rangle = \sum_a n_a^t |a\rangle \quad (7.10)$$

The restriction is required to commute with fusion. This can be written as  $B|a\rangle \times B|b\rangle = B(|a\rangle \times |b\rangle)$  or more explicitly

$$n_{\bar{a}}^t = n_a^t, \quad \sum_{t,s} n_a^t n_b^s N_{ts}^r = \sum_c N_{ab}^c n_c^r \quad (7.11)$$

We will study the behaviour of the Perron-Frobenius eigenvectors  $|\omega_{\mathcal{A}}\rangle$  and  $|\omega_{\mathcal{T}}\rangle$  under the restriction and lift. We have

$$B^\dagger|\omega_{\mathcal{T}}\rangle = \sum_{a,t} n_a^t d_t |a\rangle \quad (7.12)$$

$$= \sum_a d_a |a\rangle \quad (7.13)$$

$$= |\omega_{\mathcal{A}}\rangle \quad (7.14)$$

This holds, because quantum dimensions are conserved under the restriction.

We will now show that

$$B|\omega_{\mathcal{A}}\rangle = q|\omega_{\mathcal{T}}\rangle \quad (7.15)$$

Let us define the map  $M: \mathcal{T} \rightarrow \mathcal{T}$  by

$$M \equiv \sum_a M_a = \sum_{a,t} n_a^t N_t \quad (7.16)$$

i.e.  $M_a$  is the fusion matrix of the restriction of  $a$  and  $M$  is obtained by taking the sum over all charges  $a$  of  $\mathcal{A}$ . The matrix elements of  $M$  are all strictly positive, since for any  $s, r \in \mathcal{T}$  there is some  $t \in \mathcal{T}$  with  $N_{ts}^r > 0$  and also any  $t \in \mathcal{T}$  occurs in the decomposition of some  $a \in \mathcal{A}$ . By the Perron-Frobenius theorem, the eigenspace of  $M$  corresponding to the Perron-Frobenius eigenvalue  $\sum_a d_a$  is non-degenerate and spanned by  $|\omega_{\mathcal{T}}\rangle$ . If we can show that

$$MB|\omega_{\mathcal{A}}\rangle = \left(\sum_a d_a\right)B|\omega_{\mathcal{A}}\rangle \quad (7.17)$$

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it therefore follows that  $B|\omega_{\mathcal{A}}\rangle = \lambda|\omega_{\mathcal{T}}\rangle$  for some  $\lambda \in \mathbb{C}$ . In components, equation (7.17) reads

$$\sum_{a,c,s,r} d_c n_a^r n_c^s N_{rt}^s = \sum_{a,b} d_a d_b n_t^b \quad (7.18)$$

To prove this equation, we manipulate the left hand side using  $N_{rt}^s = N_{\bar{r}s}^t$  and  $n_a^r n_{\bar{a}}^{\bar{r}}$ , apply (7.11), use that  $N_{ac}^b = N_{ab}^c = N_{ba}^c$  and finally apply (7.8). Some intermediate results are shown in the following computation

$$\sum_{a,c,s,r} d_c n_a^r n_c^s N_{rt}^s = \sum_{a,c,s,r} d_c n_a^r n_c^s N_{rs}^t \quad (7.19)$$

$$= \sum_{a,b,c} d_c N_{ac}^b n_n^t \quad (7.20)$$

$$= \sum_{a,b,c} d_c N_{ab}^c n_b^t \quad (7.21)$$

$$= \sum_{a,b} d_a d_b n_b^t \quad (7.22)$$

Now compare components on the left and right hand side of  $B|\omega_{\mathcal{A}}\rangle = \lambda|\omega_{\mathcal{T}}\rangle$ . One finds

$$\lambda = \frac{\sum_a n_a^t d_a}{d_t} = q \quad (7.23)$$

so the fraction is indeed independent of the choice of  $t$ . The equation of the number  $q$  resembles the embedding index introduced by Dynkin in his work on the classification of subalgebras of compact Lie algebras, hence it was named the *quantum embedding index* [6].

From  $B|\omega_{\mathcal{A}}\rangle = Q|\omega_{\mathcal{T}}\rangle$  and  $B^\dagger|\omega_{\mathcal{T}}\rangle = |\omega_{\mathcal{A}}\rangle$  it follows that the integer square matrices  $B^\dagger B$  and  $BB^\dagger$  satisfy

$$B^\dagger B|\omega_{\mathcal{A}}\rangle = q|\omega_{\mathcal{A}}\rangle \quad \text{and} \quad (7.24)$$

$$BB^\dagger|\omega_{\mathcal{T}}\rangle = q|\omega_{\mathcal{T}}\rangle \quad (7.25)$$

From this we see that  $q = \mathcal{D}_{\mathcal{A}}^2/\mathcal{D}_{\mathcal{T}}^2$  because

$$\mathcal{D}_{\mathcal{A}}^2 = \langle \omega_{\mathcal{A}} | \omega_{\mathcal{A}} \rangle = \langle \omega_{\mathcal{T}} | BB^\dagger | \omega_{\mathcal{T}} \rangle = q \langle \omega_{\mathcal{T}} | \omega_{\mathcal{T}} \rangle = q \mathcal{D}_{\mathcal{T}}^2 \quad (7.26)$$

If we used (6.15), we find that

$$\frac{\mathcal{D}_{\mathcal{A}}}{\mathcal{D}_{\mathcal{T}}} = \frac{\mathcal{D}_{\mathcal{T}}}{\mathcal{D}_{\mathcal{U}}} = \sqrt{q} \quad (7.27)$$

This shows that the topological entropy of the respective phases changes following

$$\gamma_{\mathcal{A}} \xrightarrow{+\log \sqrt{q}} \gamma_{\mathcal{T}} \xrightarrow{+\log \sqrt{q}} \gamma_{\mathcal{U}} \quad (7.28)$$

## 7.2 Topological $S$ -matrix as an order parameter

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For a complete result, it would be nice to prove the relations on the total quantum dimensions. In [48], this is done by proving that  $\mathcal{U}$  is a modular tensor category when  $\mathcal{A}$  is modular, and proving that there is a kind of commutation relation for the  $S$ -matrices of the respective theories and the restriction map. If we agree to denote the  $S$ -matrix of  $\mathcal{U}$  by  $\tilde{S}_{uv}$ , where  $u, v$  run over the  $\mathcal{U}$ -labels, one has  $BS = \tilde{S}B$ , or in components

$$\sum_a n_a^u S_{ab} = \sum_u \tilde{S}_{uv} n_b^v \quad (7.29)$$

## 7.2 Topological $S$ -matrix as an order parameter

Much of the information about the topological order of a system is encoded in the topological  $S$ -matrix. Knowing the  $S$ -matrix gives direct access to the quantum dimensions of the charges, as well as the total quantum dimension, and the fusion rules can be derived via the Verlinde formula. In addition, it is important in experimental verification of topological order, as it is closely related to the monodromy matrix that often appears in the probabilities for interferometry experiments.

Using the extended formalism for topological symmetry breaking from the last chapter, we can study the effect of the formation of a Bose condensate on the  $S$ -matrix of the theory diagrammatically. We write down an explicit formula and illustrate this by calculations for the  $\mathfrak{su}(2)_4$  and  $\mathfrak{su}(2)_{10}$  example.

It will be convenient to leave the normalization factor  $\mathcal{D}^{-1}$  out of the definition of the  $S$ -matrix in this section. Hence, we define the  $S$ -matrix of  $\mathcal{A}$  simply as the evaluation of the well known Hopf link

$$S_{ab} = \left( \begin{array}{c} \text{Hopf link} \\ a \quad b \end{array} \right) \quad (7.30)$$

The monodromy equation lets us express the  $S$ -matrix in terms of the fusion multiplicities, quantum dimensions and twist factors as

$$S_{ab} = \sum_c N_{ab}^c \frac{\theta_a \theta_b}{\theta_c} d_c \quad (7.31)$$

For  $\mathfrak{su}(2)_4$ , this gives for example

$$S_{ab} = \begin{array}{c|ccccc} a, b & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & \sqrt{3} & 2 & \sqrt{3} & 1 \\ 1 & \sqrt{3} & \sqrt{3} & 0 & -\sqrt{3} & -\sqrt{3} \\ 2 & 2 & 0 & -2 & 0 & 2 \\ 3 & \sqrt{3} & -\sqrt{3} & 0 & \sqrt{3} & -\sqrt{3} \\ 4 & 1 & -\sqrt{3} & 2 & -\sqrt{3} & 1 \end{array} \quad (7.32)$$

The  $S$ -matrix describes a process where two particle-anti-particle-pairs are drawn out of the vacuum, the two particles undergo a full monodromy and the pair annihilates

## 7. INDICATORS FOR TOPOLOGICAL ORDER

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again by fusing to the vacuum. Now imagine the same process in the phase after condensation. To make sense, the charges labelling the two loops of the Hopf-link should lie in the unconfined theory  $\mathcal{U}$ . However, we can calculate the diagram for charges in  $\mathcal{T}$  just as well, when we expand the operator as a diagram in  $\mathcal{A}$ . So we will write  $\tilde{S}_{st}$  for the  $S$ -matrix elements of the broken theory and allow  $s, t \in \mathcal{T}$ . But writing down the diagram for  $\tilde{S}_{st}$  is not as straight forward as labelling the Hopf-link. Since the condensate is like the vacuum for the new theory, we can connect the two loops of the Hopf-link by a condensate line for free. At least, this makes sense physically. From the mathematics, this also appears as the right definition of the  $S$ -matrix for the broken theory. Furthermore, we need a normalization factor of  $q^2$ .

So we define  $\tilde{S}_{Tst}$  diagrammatically as

$$\tilde{S}_{Tst} = \frac{1}{q^2} \left( \text{diagram of a Hopf-link with two loops labeled } s \text{ and } t \right) \quad (7.33)$$

where  $q = d_\phi$  is the quantum embedding index. The vertices are defined following the previous chapter. The vertex involving  $t$  expands as

$$\text{vertex with } t = \sum_{\substack{\alpha \in \phi \\ a, b \in t}} {}^{(t)}A_b^a \text{ vertex with } a \quad (7.34)$$

For the vertex involving  $s$ , we need to take a complex conjugation into account

$$\text{vertex with } s = \sum_{\substack{\alpha \in \phi \\ a, b \in s}} ({}^{(s)}A_b^a)^* \text{ vertex with } b \quad (7.35)$$

Note that because the diagram for  $\tilde{S}_{st}$  closes on the top and bottom, actually no vertices with  $a \neq b$  enter the equation and we only need the coefficients  ${}^{(s)}A_a^{\alpha a}, {}^{(t)}A_a^{\alpha a}$ .

Now take  $\phi = 0 + \delta$ , as is applicable to  $\mathfrak{su}(2)_4$  and  $\mathfrak{su}(2)_{10}$ . Write  ${}^{(s, \delta)}A_b^{\delta a} = {}^{(s, \delta)}A_b^a$ , as we did before. Define

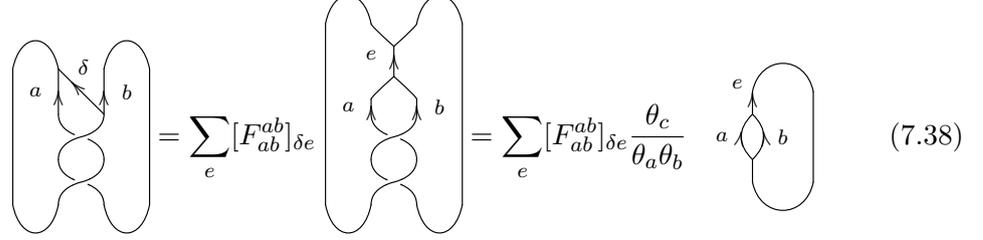
$$\bar{S}_{ab} = \left( \text{diagram of a Hopf-link with two loops labeled } a \text{ and } b \text{ and a condensate line labeled } \delta \text{ connecting them} \right) \quad (7.36)$$

Then we can write  $\tilde{S}_{st}$  explicitly as

$$\tilde{S}_{st} = \sum_{\substack{a \in s \\ b \in t}} \left( S_{ab} + ({}^{(s)}A_a^a)^* {}^{(t)}A_b^a \bar{S}_{ab} \right) \quad (7.37)$$

## 7.2 Topological $S$ -matrix as an order parameter

By using the diagrammatic formalism, we can calculate  $\bar{S}_{ab}$  in terms of  $F$ -symbols, quantum dimensions and topological spins, as follows.



This gives

$$\bar{S}_{ab} = \sum_c [F_{ab}^{ac}]_{\delta c} \frac{\theta_c}{\theta_a \theta_b} \sqrt{d_a d_b d_c} \quad (7.39)$$

In the case of  $\mathfrak{su}(2)_4$ , this gives the matrix

$$\bar{S}_{ab}^4 = \begin{array}{c|cccc} a, b & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2\sqrt{3}i & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \end{array} \quad (7.40)$$

Note that we need the splitting of fields in the phase transition for  $\bar{S}_{ab}$  to be non-trivial, because only fields  $a, b$  with  $a \in \delta \times a$  and  $b \in \delta \times b$  can give non-zero contributions, and we have seen earlier that this is equivalent to splitting. Indeed, in the above example we find only one non-zero matrix coefficient,  $\bar{S}_{22}$ , corresponding to the single splitting field 2.

Let us now calculate  $\tilde{S}_{st}$  when  $\mathcal{A} = \mathfrak{su}(2)_4$ . The quantum embedding index in this case is  $q = 2$ . We get

$$\tilde{S}_{st} = \begin{array}{c|cccc} s, t & (0+4) & (1+3) & 2_+ & 2_- \\ \hline (0+4) & 1 & 0 & 1 & 1 \\ (1+3) & 0 & 0 & 0 & 0 \\ 2_+ & 1 & 0 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 2_- & 1 & 0 & e^{-2\pi i/3} & e^{2\pi i/3} \end{array} \quad (7.41)$$

since  $e^{\pm 2\pi i/3} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$ . We recognize  $(1+3)$  as the confined charge of  $\mathcal{T}$ . The residual matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{pmatrix} \quad (7.42)$$

is indeed the unique unnormalized  $S$ -matrix describing topological order with the fusion rules found in the previous chapter [71] and belongs to the  $\mathfrak{su}(3)_1$  model consistent with the analysis of [9].

## 7. INDICATORS FOR TOPOLOGICAL ORDER

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Let us turn to the more intricate example of  $\mathfrak{su}(2)_{10}$ . Because of the size of the matrices, it is a bit inconvenient to give  $S_{ab}$  and  $\bar{S}_{ab}$ , but the calculations are straight forward. They are printed in the appendix. The result for  $\tilde{S}_{st}$  is

$$\tilde{S}_{st} =$$

$s, t$	(0 + 6)	(1 + 5 + 7)	(2 + 4 + 6 + 8)	(3 + 5 + 9)	(3 + 7)	(4 + 10)
(0 + 6)	1	0	0	0	$\sqrt{2}$	1
(1 + 5 + 7)	0	0	0	0	0	0
(2 + 4 + 6 + 8)	0	0	0	0	0	0
(3 + 5 + 9)	0	0	0	0	0	0
(3 + 7)	$\sqrt{2}$	0	0	0	0	$-\sqrt{2}$
(4 + 10)	1	0	0	0	$-\sqrt{2}$	1

Again, the confined excitations show as rows and columns of zeros. The  $S$ -matrix for the bulk theory is

$$\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \quad (7.43)$$

which is indeed the  $S$ -matrix for  $\mathfrak{so}(5)_1$ , in correspondence with the results of [9].

We see that the formalism developed in the previous chapter allows us to treat the phase transition directly on the level of the  $S$ -matrix. We find the confined charges as rows and columns of zeros. Since the  $S$ -matrix encodes most of the interesting topological order data, it is perfectly fit as an order parameter for topological order.

An interesting feature of the present approach is that it allows for an intermediate treatment. We might calculate several variations of equation (7.37), for instance, we can calculate

$$S_{ab} + \bar{S}_{ab} \quad (7.44)$$

possibly with the insertion of the  $A$ 's. It appears that in lattice simulations of DGTs, where the parameters are varied, phase transitions occur that can be signalled by string-like operators (see [13]) essentially measuring the condensed excitations. Also,  $S$ -matrix elements can be measured in these simulations. It appears that  $\bar{S}$ -diagrams start to contribute when a non-trivial condensate is formed, but since the operators that are measured are labelled by the sectors of the original phase one seems to find (7.44) or a closely related quantity.

Further development of both the numeric and analytic approaches to these systems would be needed to obtain a complete understanding. This thesis presents the first explorations of a diagrammatic formalism for topological symmetry breaking. In appendix B, we outline how we believe general diagrams for the broken phase can be constructed. The relation to lattice models provides an interesting direction for future research.

## 7.2 Topological $S$ -matrix as an order parameter

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$q=2$

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## Conclusion and outlook

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We have discussed topological order for planar systems, and in particular the role of quantum groups and modular tensor categories. A general graphical formalism for topologically ordered phases based on tensor categories was discussed. As discussed in chapter 4, the essential data is given by fusion coefficients  $N_{ab}^c$ ,  $F$ -symbols  $[F_d^{abc}]_{ef}$  and  $R$ -symbols  $R_c^{ab}$ . We showed how to calculate these for the quantum double theories without multiplicities and the important  $\mathfrak{su}(2)_k$  series in chapter.

Making use of the literature on tensor categories, we have reconsidered topological symmetry breaking phase transitions and argued that Bose condensates, in this setting, correspond to commutative algebra objects. A diagrammatic formalism for topological symmetry breaking was subsequently explored. The necessary consistency conditions were solved explicitly for representative examples.

We derived some new information regarding topological symmetry breaking in this thesis. The particle spectrum of broken theory can be derived by calculating  $\phi \times a$ . It was also shown that the quantum dimension of the condensate is a universal number  $q$ , the quantum embedding index, that characterizes the phase transitions and in fact relates to the topological entanglement entropy.

Using the extended formalism for topological symmetry breaking, we could study the  $S$ -matrix analytically before and after the phase transition. We recovered the results of the breaking of  $\mathfrak{su}(2)_4$  and  $\mathfrak{su}(2)_{10}$ . This suggests the usefulness of the  $S$ -matrix as an order parameter for topological order.

The link that exists between commutative algebras in braided tensor categories and Bose condensates suggests that this structure should appear in a variety of physical contexts. If it, indeed, turns out that topological symmetry breaking is the leading underlying mechanism for phase transitions between planar topologically ordered phases, this gives commutative algebras in braided tensor categories a role analogous to the role of subgroups in the regular theory of symmetry breaking phase transitions. Fur-

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ther explorations in this direction are an interesting endeavour and should definitely be continued.

One can imagine several directions. For instance, further application in addressing questions physical contexts such as the FQHE and topological quantum computation are likely to lead to interesting insights. Using the graphical formalism, it should for example be possible to discuss questions concerning interferometry measurements for systems with multiple phases.

Furthermore, relating the results to lattice simulations that calculate can further reveal the significance and meaning of the coefficients that were needed to solve the equations that resulted from the consistency conditions of the condensate and charge vertices. Many questions remain in this area.

While we believe that the theory of algebras in tensor categories provides a constructive way to construct the category of the broken phase, the formalism can still be refined. For example, how do we obtain the fusion multiplicities in a more direct manner? To clarify such questions, it is useful to further study and translate the mathematical results from the tensor category literature to a physical context. We think that inspecting notions that play an important role in the mathematical literature and assessing their physical relevance will lead to a considerable amount of interesting and useful work. An example might be the center construction, which produces a commutative algebra in the center (quantum double) of the category from any non-commutative algebra. This could relate to phase transitions in the Levin-Wen models, as the category that governs excitations is the center (double) of the category that is used as input in these models.

Finally, either from a mathematicians of physicists perspective, it is interesting to identify the results from the tensor category approach in alternative formulations. For instance, one could study topological quantum field theory, or search for relations to questions in low-dimensional topology.

In conclusion, a considerable amount of interesting and useful work lies ahead to be undertaken.

## APPENDIX A

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### Quasi-triangular Hopf algebras

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Below is a gathering of the axioms of quasi-triangular Hopf algebras (or quantum groups) with the corresponding mathematical nomenclature. We refer to [46] and [75] for a thorough treatment.

A Hopf algebra  $\mathcal{H}$  is, in short, a bialgebra with an antipode. A bit more elaborate, this means that we have an associative algebra with multiplication  $m: \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$  and unit  $1: \mathbb{C} \rightarrow \mathcal{H}$ . Associativity is given by the condition

$$m(m \otimes \text{id}) = m(\text{id} \otimes m) \tag{A.1}$$

The unit axiom is

$$m(1 \otimes \text{id}) = m(\text{id} \otimes 1) = \text{id} \tag{A.2}$$

Here  $\text{id}$  is the identity on  $\mathcal{H}$ .

It is also required that  $\mathcal{H}$  is a coalgebra, which requires the definition of a comultiplication

$$\Delta: H \rightarrow H \otimes H \tag{A.3}$$

and a counit

$$\epsilon: H \rightarrow \mathbb{C} \tag{A.4}$$

They have to satisfy the coassociativity axiom

$$(\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta \tag{A.5}$$

and the counit property

$$(\epsilon \otimes \text{id})\Delta = \text{id} = (\text{id} \otimes \epsilon)\Delta \tag{A.6}$$

There is a natural multiplication, also denoted  $m$ , on  $\mathcal{H} \otimes \mathcal{H}$ . For elements  $a_1 \otimes a_2$  and  $b_1 \otimes b_2$  of  $\mathcal{H} \otimes \mathcal{H}$ , this is  $m(a_1 \otimes a_2, b_1 \otimes b_2) = m(a_1, b_1) \otimes m(a_2, b_2)$ . An algebra that is also

## A. QUASI-TRIANGULAR HOPF ALGEBRAS

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a coalgebra, is called a bialgebra if the algebra and coalgebra structure are compatible. Compatibility can be formulated as the requirement that the comultiplication and counit are algebra homomorphisms (i.e, commute with the appropriate multiplication).

One ingredient is still lacking to make  $\mathcal{H}$  a Hopf algebra, namely the existence of an antipode. This is a linear map  $S: \mathcal{H} \rightarrow \mathcal{H}$  satisfying

$$m(\text{id} \otimes S)\Delta = 1\epsilon = m(S \otimes \text{id})\Delta \quad (\text{A.7})$$

It can be shown that, if a bialgebra admits an antipode, it is unique. So there is at most one Hopf algebra structure on any bialgebra.

Now what is the significance of Hopf algebras to physics? We are of course interested in the representations of  $H$  to describe the particle spectrum in some (2+1)d quantum theory. The comultiplication gives the definition of tensor product representations needed to describe multi-particle states. Suppose  $\alpha, \beta$  are two representations of  $\mathcal{H}$  with carrier spaces  $V_\alpha, V_\beta$  given by  $\Pi^{\alpha, \beta}: H \rightarrow \text{End}(V_{\alpha, \beta})$ . Then we define the tensor product representation  $\alpha \otimes \beta$ , acting on  $V_\alpha \otimes V_\beta$ , by

$$\Pi^{\alpha \otimes \beta}(a) \equiv \Pi^\alpha \otimes \Pi^\beta(\Delta(a)), \quad a \in \mathcal{H} \quad (\text{A.8})$$

Coassociativity ensures that multi-particle states defined by successive application of the coproduct do not depend on the chosen order for the application of the comultiplication. The tensor product representations will generally fall apart as direct sums of irreducible representations. This gives rise to the fusion rules of the theory.

The counit gives the definition of the trivial representation  $\Pi^0 = \epsilon$ . The counit property precisely provides the triviality of fusion on the level of representations,  $0 \otimes \alpha = \alpha = \alpha \otimes 0$ , because

$$\Pi^{0 \otimes \alpha}(a) = (\epsilon \otimes \Pi^\alpha)\Delta(a) = \Pi^\alpha(\epsilon \otimes \text{id})\Delta(a) = \Pi^\alpha(a) \quad (\text{A.9})$$

The notion of anti-particles or conjugate charges (dual representations) is implemented using the antipode. Let us denote the conjugate (or dual) representation of  $\alpha$  by  $\bar{\alpha}$ . We can define such a representation on the dual vector space  $V_\alpha^*$  by means of the antipode as

$$\Pi^{\bar{\alpha}}(a) = \Pi^\alpha(S(a))^T \quad (\text{A.10})$$

where the  $T$  denotes matrix transposition.

The ingredient needed to describe the braiding of representations of a Hopf algebra is the universal  $R$ -matrix. This is an invertible element  $R \in \mathcal{H} \otimes \mathcal{H}$  which has the properties

$$\Delta^{\text{op}}(a)R = R\Delta(a) \quad (\forall a \in \mathcal{H}) \quad (\text{A.11})$$

$$(\Delta \otimes \text{id})(R) = R_{13}R_{23} \quad (\text{A.12})$$

$$(\text{id} \otimes \Delta)(R) = R_{13}R_{12} \quad (\text{A.13})$$

Here  $\Delta^{\text{op}} = \sigma\Delta$  is the composition of the comultiplication with the exchange  $\sigma$  of tensor factors in  $\mathcal{H} \otimes \mathcal{H}$ . The notation  $R_{ij}$  is an abbreviation for the action of  $R$  on the  $i^{\text{th}}$

and  $j^{\text{th}}$  tensor leg of  $\mathcal{H}^{\otimes 3}$ , so for example  $R_{12} = R \otimes 1$ . To implement the clockwise exchange two particles, we act by  $R$  on the tensor product representation, and then exchange the tensor legs. So for example, when we have a system of three particles all carrying representation  $\alpha$ , the exchange of the left two particles is effectuated by applying

$$(\tau \otimes \text{id})(\Pi^\alpha \otimes \Pi^\alpha \otimes \Pi^\alpha)(R \otimes 1) \quad (\text{A.14})$$

where  $\tau$  denotes the flip of tensor product legs  $\tau|v\rangle|w\rangle = |w\rangle|v\rangle$ .

The defining properties of  $R$  ensure that exchanging particles by application of  $\tau R$  makes physical sense. Using either A.11 and A.12 or A.11 and A.13 one may prove the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (\text{A.15})$$

It implies that in any representation of  $\mathcal{H}$  we have the equality

$$(\tau R \otimes 1)(1 \otimes \tau R)(\tau R \otimes 1) = (1 \otimes \tau R)(\tau R \otimes 1)(1 \otimes \tau R) \quad (\text{A.16})$$

from which one may deduce that in a system of  $n$  identical particles, the exchange of adjacent particle satisfies the defining relations for the braid group. Since  $R$  is invertible, this generates a representation of the braid group  $B_n$ . Because the exchange commutes with the action of  $\mathcal{H}$ , it follows that the system carries a representation of  $\mathcal{H} \times B_n$

A Hopf algebra  $\mathcal{H}$  with universal  $R$ -matrix is called *quasitriangular*, so this is what we understand by the term quantum group.

In relation to the graphical formalism for topological symmetry breaking treated in chapter 6 it is interesting to represent the Hopf algebra axioms in a diagrammatic fashion. This elucidates why the definition of the condensate is known to mathematicians as an commutative algebra, or actually coalgebra, in the context of braided tensor categories.

Represent the identity by a vertical line, the multiplication by an unlabelled fusion vertex and the comultiplication by a similar splitting vertex. Thus maps are displayed by lines from bottom to top, endpoints corresponding to a copy of  $\mathcal{H}$  and the tensor product is given by juxtaposition. The unit and counit will be represented by a line starting or ending at a dot, which is consistent with the interpretation of  $\mathbb{C}$  as the vacuum module by means of  $\epsilon$ . Thus

$$\begin{array}{ccccc}
 \begin{array}{c} | \\ \text{id} \end{array} & 
 \begin{array}{c} \diagup \\ | \\ \diagdown \\ m \end{array} & 
 \begin{array}{c} \diagdown \\ | \\ \diagup \\ \Delta \end{array} & 
 \begin{array}{c} | \\ \bullet \\ 1 \end{array} & 
 \begin{array}{c} \bullet \\ | \\ \epsilon \end{array}
 \end{array} \quad (\text{A.17})$$

The unit and counit axiom then become

$$\begin{array}{c} \bullet \\ | \\ \diagup \\ \diagdown \end{array} = \begin{array}{c} | \\ \bullet \end{array} = \begin{array}{c} \diagup \\ | \\ \diagdown \\ \bullet \end{array} \quad \text{and} \quad \begin{array}{c} \diagdown \\ | \\ \diagup \\ \bullet \end{array} = \begin{array}{c} | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \diagdown \\ \diagup \end{array} \quad (\text{A.18})$$

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respectively. Similarly we can depict (co)associativity graphically. The associativity axiom becomes

$$\begin{array}{c} | \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ | \end{array} = \begin{array}{c} | \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ | \end{array} \quad (\text{A.19})$$

while coassociativity reads

$$\begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ | \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \\ | \end{array} \quad (\text{A.20})$$

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## Tensor categories: from math to physics

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We present a sketch of the definitions of a tensor category and the relevant additional structure that underlies the main body of the text. See [54] for the standard introduction to category theory. See [14] or [75] for a thorough treatment of tensor categories and the mathematics of modular functors, modular tensor categories and TQFT. A discussion of tensor categories in relation to quantum groups is also included in [46]. An influential paper on the relation of tensor categories to invariants of knots and three-manifolds is [70] which also includes a concise and clear recap of tensor categories. We outline the formalities such that it becomes clear how they relate to the main body of the text. For a thorough treatment, we must refer to the literature.

### B.1 Objects, morphisms and functors

Much of mathematics can be understood in terms of properties of the structure preserving maps, such as homomorphisms of groups, acting between objects of a certain class, e.g. groups, leaving out the internal structure of the object. This is the central idea of a category. Examples are the category of topological spaces with homeomorphisms, the category of sets and maps, the category of groups and group homomorphisms, the category of 3-dimensional manifolds and smooth maps and the category of vector spaces with linear maps.

A category  $\mathcal{A}$  consists of a class  $\text{Ob}(\mathcal{A})$ , elements of which are called the *objects* of  $\mathcal{A}$ , and a class  $\text{Hom}(\mathcal{A})$ , elements of which are called the *morphisms* of  $\mathcal{A}$ . Any morphism  $f$  has a source  $A \in \text{Ob}(\mathcal{A})$  and a target  $B \in \text{Ob}(\mathcal{A})$  and is often denoted by an arrow  $f: A \rightarrow B$  or

$$A \xrightarrow{f} B \tag{B.1}$$

The morphisms from  $A$  to  $B$  are also denoted by  $\text{Hom}_{\mathcal{A}}(A, B)$ , or simply  $\text{Hom}(A, B)$ .

## B. TENSOR CATEGORIES: FROM MATH TO PHYSICS

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If  $f \in \text{Hom}(A, B)$  and  $g \in \text{Hom}(B, C)$  we can compose the morphisms  $f$  and  $g$ . Composition is denoted  $gf$  (often  $g \circ f$ ). In a diagram, we have the corresponding composition of arrows

$$gf: A \xrightarrow{f} B \xrightarrow{g} C \quad (\text{B.2})$$

For any object  $A$  there is a special morphism called the identity  $\text{id}_A$  which obeys

$$\text{id}_A f = f = f \text{id}_A, \quad \forall f \in \text{Hom}(A, A) \quad (\text{B.3})$$

A morphism  $f: A \rightarrow B$  is called an *isomorphism* if there exists an morphism  $f^{-1}: B \rightarrow A$  such that

$$f^{-1}f = \text{id}_A, \quad ff^{-1} = \text{id}_B, \quad (\text{B.4})$$

The right notion for maps between categories is that of a functor. A functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  consists of a map  $F: \text{Ob}(\mathcal{A}) \rightarrow \text{Ob}(\mathcal{B})$  and a map  $F: \text{Hom}(\mathcal{A}) \rightarrow \text{Hom}(\mathcal{B})$  such that

- $F(\text{id}_A) = F_{\text{id}_{F(A)}}$  for all  $A \in \text{Ob}(\mathcal{A})$
- if  $f: A \rightarrow B$  then  $F(f): F(A) \rightarrow F(B)$
- if  $f$  and  $g$  are morphism of  $\mathcal{A}$  then

$$F(fg) = F(f)F(g) \quad (\text{B.5})$$

From two categories  $\mathcal{A}$  and  $\mathcal{B}$ , one can construct the product category  $\mathcal{A} \times \mathcal{B}$ . If  $\mathcal{A}, \mathcal{B}$  describe physical systems, this corresponds to the combined system when they do not interact. Objects of  $\mathcal{A} \times \mathcal{B}$  are pairs of objects  $(A, B)$ , with  $A \in \text{Ob}(\mathcal{A}), B \in \text{Ob}(\mathcal{B})$ , and morphisms are just pairs of morphisms, or written formally

$$\text{Hom}_{\mathcal{A} \times \mathcal{B}}((A, B), (A', B')) = \text{Hom}_{\mathcal{A}}(A, A') \times \text{Hom}_{\mathcal{B}}(B, B') \quad (\text{B.6})$$

If  $F$  and  $G$  are two functors and  $F: \mathcal{A} \rightarrow \mathcal{B}, G: \mathcal{B} \rightarrow \mathcal{A}$ , a *natural transformation*  $\alpha: F \rightarrow G$  is a family of morphisms  $\alpha_A: F(A) \rightarrow G(A)$  such that the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ F(f) \downarrow & & \downarrow G(f) \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array} \quad (\text{B.7})$$

commutes. If  $\alpha_A$  is an isomorphism for every  $A$ ,  $\alpha$  is called a *natural isomorphism*.

There is a notion of equivalence for categories. Two categories  $\mathcal{A}$  and  $\mathcal{B}$  are equivalent if there are functors  $F: \mathcal{A} \rightarrow \mathcal{B}$  and  $G: \mathcal{B} \rightarrow \mathcal{A}$  such that the compositions  $FG$  and  $GF$  are naturally isomorphic to the respective identity functors.

We are interested in categories to model anyons. To regard certain objects as anyons and morphisms as anyon operators, these categories should incorporate fusion and braiding. Fusion requires the definition of a tensor product and direct sum in these categories. We will outline the definitions below.

## B.2 Direct sum

Categories that have a natural notion of direct sum are known as Abelian or linear categories (depending on the properties taken along in the definition). For our purposes,  $\mathbb{C}$ -linear categories give the appropriate notion.

In a  $\mathbb{C}$ -linear category, spaces of morphism are required to be complex vector spaces. This basically means that there is a zero morphism  $0 \in \text{Hom}(A, B)$  and we can take linear combinations of morphisms. The composition of morphisms is bilinear  $(\lambda f)(\mu g) = \lambda\mu fg$ , where  $f, g$  are composable morphisms and  $\lambda, \mu \in \mathbb{C}$ .

One must be able to take direct sums of objects as well and “define”  $A = A_1 \oplus A_2$ . This is not really a definition, rather an object  $A$  must exist with projection and inclusion morphisms for the summands

$$p_j: A \rightarrow A_j, \quad i_j: A_j \rightarrow A, \quad j = 1, 2 \tag{B.8}$$

such that

$$p_j i_j = \text{id}_{A_j}, \quad \text{and} \quad i_1 p_1 + i_2 p_2 = \text{id}_A \tag{B.9}$$

For the spaces of morphisms one has

$$\text{Hom}(\oplus_j A_j, \oplus_k B_k) = \bigoplus_{j,k} \text{Hom}(A_j, B_k) \tag{B.10}$$

Let us represent morphisms  $f: A \rightarrow B$  as a diagram

$$\begin{array}{c} B \\ | \\ \boxed{f} \\ | \\ A \end{array} \tag{B.11}$$

then (B.10) says that we can write

$$\begin{array}{c} \oplus_k B_k \\ | \\ \boxed{f} \\ | \\ \oplus_j A_j \end{array} = \sum_{j,k} \begin{array}{c} B_k \\ | \\ \boxed{f_{j,k}} \\ | \\ A_j \end{array} \tag{B.12}$$

Simple objects are defined as objects for which the endomorphism space is one-dimensional,

$$A \text{ simple} \iff \text{Hom}(A, A) = \mathbb{C} \text{id}_A \tag{B.13}$$

## B. TENSOR CATEGORIES: FROM MATH TO PHYSICS

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The category is called *semi-simple* if every object is isomorphic to a sum of simple objects. We shall write simple object with lower case letters, as these correspond to the particles in physical theories. So any object  $A$  has

$$A \simeq \oplus_j a_j, \quad a_j \text{ simple} \quad (\text{B.14})$$

### B.3 Tensor categories

The tensor product in a category is an important notion for us, as it gives the fusion of quantum numbers. The unit object should corresponds to the vacuum in this context. We give the definitions below.

Let  $\mathcal{A}$  be a category. A *tensor product*  $\otimes$  is a functor  $\otimes: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ . Hence, for any pair of objects  $(A, B) \in \text{Ob}(\mathcal{A}) \times \mathcal{A}$  we get an object  $A \otimes B$  of  $\mathcal{A}$ . Similarly, any pair of  $\mathcal{A}$ -morphisms  $(f, g)$  gives a morphism  $f \otimes g \in \text{Hom}(\mathcal{A})$ . The fact that  $\otimes$  is required to be a functor implies

$$(f' \otimes g')(f \otimes g) = f'f \otimes g'g \quad (\text{B.15})$$

and

$$\text{id}_{A \otimes B} = \text{id}_A \otimes \text{id}_B \quad (\text{B.16})$$

An *associativity constraint* for  $\otimes$  is a natural isomorphism

$$\alpha: \otimes(\otimes \times \text{id}) \rightarrow \otimes(\text{id} \times \otimes) \quad (\text{B.17})$$

This means that for any triple  $(A, B, C)$  of objects of  $\mathcal{A}$ , there is an isomorphism

$$\alpha_{A,B,C}: (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C) \quad (\text{B.18})$$

such that

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) \\ (f \otimes g) \otimes h \downarrow & & \downarrow f \otimes (g \otimes h) \\ (A' \otimes B') \otimes C' & \xrightarrow{\alpha_{A',B',C'}} & A' \otimes (B' \otimes C') \end{array} \quad (\text{B.19})$$

commutes for any morphisms  $f, g, h$  of  $\mathcal{A}$ . The associativity constraint is required to satisfy the *Pentagon axiom*, i.e. the diagrams

$$\begin{array}{ccccc} & & (A \otimes B) \otimes (C \otimes D) & & \\ & \nearrow^{\alpha_{A \otimes B, C, D}} & & \searrow^{\alpha_{A, B, C \otimes D}} & \\ ((A \otimes B) \otimes C) \otimes D & & & & A \otimes (B \otimes (C \otimes D)) \\ & \searrow_{\alpha_{A, B, C} \otimes \text{id}_D} & & \nearrow_{\text{id}_A \otimes \alpha_{B, C, D}} & \\ & & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A, B \otimes C, D}} & A \otimes ((B \otimes C) \otimes D) \end{array} \quad (\text{B.20})$$

should commute for all objects  $A, B, C, D$ .

### B.3.1 Unit object

The vacuum in the theories we considered is formally called a unit object. This is an object  $\mathbf{1} \in \text{Ob}(\mathcal{A})$  with *left* and *right unit constraint* denoted  $\ell$  and  $r$  respectively. These are collections of isomorphisms

$$\ell_A: \mathbf{1} \otimes A \rightarrow A, \quad r_A: A \otimes \mathbf{1} \rightarrow A \quad (\text{B.21})$$

such that

$$\begin{array}{ccc} \mathbf{1} \otimes A & \xrightarrow{\ell_A} & A \\ \text{id}_{\mathbf{1}} \otimes f \downarrow & & \downarrow f \\ \mathbf{1} \otimes B & \xrightarrow{\ell_B} & B \end{array} \quad (\text{B.22})$$

The unit constraints have to satisfy the *Triangle axiom*, which requires the commutativity of the following diagram

$$\begin{array}{ccc} (A \otimes \mathbf{1}) \otimes B & \xrightarrow{\alpha_{A, \mathbf{1}, B}} & A \otimes (\mathbf{1} \otimes B) \\ & \searrow r_A \otimes \text{id}_B & \swarrow \text{id}_A \otimes \ell_B \\ & A \otimes B & \end{array} \quad (\text{B.23})$$

We also need an isomorphism  $u: \mathbf{1} \rightarrow \mathbf{1} \otimes \mathbf{1}$  that satisfies

$$\begin{array}{ccccc} & & \mathbf{1} \otimes \mathbf{1} & & \\ & u \nearrow & & \searrow \text{id}_{\mathbf{1}} \otimes u & \\ \mathbf{1} & & & & \mathbf{1} \otimes (\mathbf{1} \otimes \mathbf{1}) \\ & u \searrow & & \nearrow \alpha_{\mathbf{1}, \mathbf{1}, \mathbf{1}} & \\ & & \mathbf{1} \otimes \mathbf{1} & \xrightarrow{u \otimes \text{id}_{\mathbf{1}}} & (\mathbf{1} \otimes \mathbf{1}) \otimes \mathbf{1} \end{array} \quad (\text{B.24})$$

A tensor category  $(\mathcal{A}, \otimes, \mathbf{1}, \alpha, \ell, r)$  is a category  $\mathcal{A}$  with tensor product  $\otimes$ , associativity constraint  $\alpha$ , unit  $\mathbf{1}$  and left and right unit constraints  $\ell$  and  $r$  such that the Pentagon axiom and Triangle axiom are satisfied.

### B.3.2 Strictness and coherence

From the definition of the associativity constraint one sees that great care should be taken in keeping track of parentheses. This could be very tedious business. The notion of a strict tensor category, basically resolves this issue.

A tensor category  $(\mathcal{A}, \otimes, \mathbf{1}, \alpha, \ell, r)$  is called strict if  $\alpha, \ell$  and  $r$  are identities of the category. We have assumed this property in the main body of the text throughout. MacLane’s coherence theorem assures that we can always do this.

One can associate a strict tensor category  $\mathcal{A}^{\text{str}}$  to any tensor category  $\mathcal{A}$ . The objects of  $\mathcal{A}^{\text{str}}$  are finite sequences of objects of objects of  $\mathcal{C}$ , i.e. are of the form  $(A_1, \dots, A_k)$  with  $A_i \in \text{Ob}(\mathcal{A})$ . Morphisms between  $(A_1, \dots, A_k)$  and  $(B_1, \dots, B_{k'})$  are by definition morphisms

$$f: (\dots (A_1 \otimes A_2) \otimes \dots \otimes A_{k-1}) \otimes A_k \longrightarrow (\dots (B_1 \otimes B_2) \otimes \dots \otimes B_{k'-1}) \otimes B_{k'}$$

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of  $\mathcal{A}$ . The tensor product of  $\mathcal{A}^{\text{str}}$  is defined by

$$(A_1, \dots, A_k) \otimes (B_1, \dots, B_{k'}) = (A_1, \dots, A_k, B_1, \dots, B_{k'}) \quad (\text{B.25})$$

The unit object is  $\mathbf{1} = \emptyset$ , the empty sequence ( $k = 0$ ).

MacLanes coherence theorem [54] states that any sequence of associativity constraints to pass from one way of putting brackets in some large tensor product of objects to another way of putting brackets gives the same morphism. The easiest case is governed by the Pentagon axiom, and indeed this axiom is exactly what is needed for the general theorem. MacLanes coherence theorem implies that  $\mathcal{A}$  and  $\mathcal{A}^{\text{str}}$  are equivalent as categories. Thus there is no loss of generality in considering only strict categories.

### B.3.3 Fusion rules

The direct sum and tensor product together give rise to the fusion rules. We will consider categories with finitely many isomorphism classes of simple objects. These correspond to the finite set of ‘‘charges’’  $C_{\mathcal{A}} = \{a, b, c, \dots\}$  discussed in chapter 4. Actually, we must distinguish between simple objects and the isomorphism classes. Let  $\{V_a\}$  be a complete set of isomorphism representatives indexed by a finite set  $C_{\mathcal{A}}$ . Formally, one may say that  $a$  corresponds to the whole class of objects isomorphic to  $V_a$ . Since any tensor product of two objects is isomorphic to a direct sum of simple objects, we have

$$V_a \otimes V_b \simeq \bigoplus_c V_c^{\oplus N_{ab}^c} \quad (\text{B.26})$$

Note that

$$N_{ab}^c = \dim \text{Hom}(a \otimes b, c) = \dim \text{Hom}(c, a \otimes b) \quad (\text{B.27})$$

This gives rise to the fusion rules

$$a \times b = \sum_c N_{ab}^c \quad (\text{B.28})$$

which is basically the definition of a product on the fusion algebra  $\mathbb{C}C_{\mathcal{A}} = \bigoplus_{a \in C_{\mathcal{A}}} \mathbb{C}a$ . We will not really distinguish  $a$  and  $V_a$ .

### B.3.4 Diagrams

It is useful to adopt a diagrammatic notation for morphisms. We made extensive use of this in the main body of the text. In (B.12) a prototypical example is shown. For this, we assume that the category is strict, so there are no brackets in tensor products and we simply write  $A \otimes B \otimes C$ . The rules are as follows.

The morphisms should be read from bottom to top. If we wish to depict a morphism from  $A$  to  $B$ , then we put  $A$  at the bottom of the diagram, and  $B$  on top. Tensor products of objects correspond to sequences of objects. The identity morphism of an object  $A$  is denoted by a line segment coloured by  $A$ . A morphism  $f: A \rightarrow B$  is denoted

by a coupon with  $f$  in it. Lines coming in on the bottom correspond to the source for the morphism, lines coming out to the target. Variations on the notations of course occur.

For example, a morphism  $\mu: c \rightarrow a \otimes b$  can be denoted by a trivalent vertex labelled by  $\mu$ , and  $a, b, c$  labelling the appropriate outer legs,

$$\begin{array}{c}
 a \quad b \\
 \diagdown \quad / \\
 \mu \\
 \diagup \quad \diagdown \\
 c
 \end{array}
 \tag{B.29}$$

We recognise the splitting vertex. Indeed, let us choose a basis  $\{\mu\}$  in the vector space  $V_c^{ab} \equiv \text{Hom}(c, a \otimes b)$  for all triples of simple objects  $(a, b, c)$ . This corresponds to the definition of fusion vertices. Now consider the vector space  $V_d^{abc} \equiv \text{Hom}(d, a \otimes b \otimes c)$ . Elements of this space can be constructed via composition of morphisms  $d \rightarrow e \otimes c \rightarrow a \otimes b \otimes c$  or  $d \rightarrow a \otimes f \otimes a \otimes b \otimes c$ . This gives rise to two bases  $\{(\alpha \otimes \text{id}_c)\beta\}$  and  $\{(\text{id}_a \otimes \mu)\nu\}$  of  $V_d^{abc}$ , which can be depicted as

$$\begin{array}{c}
 a \quad b \quad c \\
 \diagdown \quad / \quad / \\
 \alpha \quad e \quad \beta \\
 \diagup \quad \diagdown \quad \diagup \\
 d
 \end{array},
 \quad
 \begin{array}{c}
 a \quad b \quad c \\
 \diagdown \quad / \quad / \\
 \nu \quad f \quad \mu \\
 \diagup \quad \diagdown \quad \diagup \\
 d
 \end{array}
 \tag{B.30}$$

These are related by a base transformation  $F$ , which gives rise to the  $F$ -moves

$$\begin{array}{c}
 a \quad b \quad c \\
 \diagdown \quad / \quad / \\
 \alpha \quad e \quad \beta \\
 \diagup \quad \diagdown \quad \diagup \\
 d
 \end{array}
 =
 \sum_{f, \mu, \nu} [F_d^{abc}]_{(e, \alpha, \beta)(f, \mu, \nu)}
 \begin{array}{c}
 a \quad b \quad c \\
 \diagdown \quad / \quad / \\
 \nu \quad f \quad \mu \\
 \diagup \quad \diagdown \quad \diagup \\
 d
 \end{array}
 \tag{B.31}$$

that played a major role in this thesis. Note that these are an incarnation of the associativity constraint for strict tensor categories (even though this constraint is trivial by definition).

The use of diagrams to represent morphism in categories is a powerful tool. Formally, one can prove that there is a functor from a suitable category of diagrams into the kind of category under consideration. The category of diagrams is in a sense universal for the kind of categories under consideration. The functor is usually called the Reshetikhin-Turaev functor. This functor can be used to produce invariants of links from, for example, modular tensor categories. Any link diagram is regarded as a morphism  $\mathbb{C} \rightarrow \mathbb{C}$  in this category, which is just a complex number.

## B.4 Duals

Anti-particles correspond on the categorical level to the notions of rigidity and dual objects.

An object  $A$  is called rigid if there is an object  $A^*$  and morphisms

$$i_A: \mathbf{1} \rightarrow A \otimes A^*, \quad e_A: A^* \otimes A \rightarrow \mathbf{1}
 \tag{B.32}$$

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(where  $i$  is for inclusion and  $e$  for evaluation) for which the compositions

$$\begin{aligned}
 A &\xrightarrow{\ell_A^{-1}} \mathbf{1} \otimes A \xrightarrow{i_A \otimes \text{id}_A} (A \otimes A^*) \otimes A \xrightarrow{\alpha_{A,A^*,A}} A \otimes (A^* \otimes A) \xrightarrow{\text{id}_A \otimes e_A} A \otimes \mathbf{1} \xrightarrow{r} A \\
 A^* &\xrightarrow{r_A^{-1}} A^* \otimes \mathbf{1} \xrightarrow{\text{id}_{A^*} \otimes i_A} A^* \otimes (A \otimes A^*) \xrightarrow{\alpha_{A^*,A,A^*}^{-1}} (A^* \otimes A) \otimes A^* \xrightarrow{e_A \otimes \text{id}_{A^*}} \mathbf{1} \otimes A^* \xrightarrow{\ell} A
 \end{aligned}$$

are identity morphisms. The object  $A^*$  is called the right dual of  $A$ . The category  $\mathcal{A}$  is called rigid if every object has a right dual.

If the category is strict, the duality axioms reduce to the statement that the following morphisms are identities

$$\begin{aligned}
 A &\xrightarrow{i_A \otimes \text{id}_A} A \otimes A^* \otimes A \xrightarrow{\text{id}_A \otimes e_A} A \\
 A^* &\xrightarrow{\text{id}_{A^*} \otimes i_A} A^* \otimes A \otimes A^* \xrightarrow{e_A \otimes \text{id}_{A^*}} A^*
 \end{aligned} \tag{B.33}$$

If we agree to denote  $i_A$  by a cup and  $e_A$  by a cap

$$i_A = A \begin{array}{c} \frown \\ \triangleright \end{array} A^*, \quad e_A = A^* \begin{array}{c} \triangleleft \\ \smile \end{array} A \tag{B.34}$$

we can incorporate these morphisms in diagrams. The duality axioms can now be expressed graphically

$$\begin{array}{c} \frown \\ \triangleright \end{array} \begin{array}{c} \triangleleft \\ \smile \end{array} = \left| \begin{array}{c} \\ \\ \\ \end{array} \right|_A, \quad \begin{array}{c} \triangleleft \\ \smile \end{array} \begin{array}{c} \frown \\ \triangleright \end{array} = \left| \begin{array}{c} \\ \\ \\ \end{array} \right|_{A^*} \tag{B.35}$$

We have used the somewhat peculiar notation with arrows in reversed orientation for  $i_A$  and  $e_A$ , which is not common but agrees with the main body of the text where it is convenient. Usually, the lines are oriented, a downward orientation corresponding to  $A$  and an upward orientation corresponding to  $A^*$ .

Note that the identity of the unit  $\text{id}_1$  is left invisible in the diagrams, which is appropriate in strict categories. In non-strict categories one can still leave them out, since there is a canonical way to put them in as expressed by MacLane's coherence theorem. But if we think about the diagrammatic calculations as done throughout this thesis, one should note that the natural isomorphisms  $r, \ell$  and  $\alpha$  should be taken into account when the unit object (vacuum) is non-trivial, as is the case when there is a non-trivial condensate.

The definition of a dual can also be formulated for left duals (replace  $A$  and  $A^*$  in the definitions of  $i_A$  and  $e_A$ ). For the categories that underlie anyon theories, right and left duals are isomorphic. The difference between left and right duality morphisms, however, lead to the Frobenius-Schur indicator.

An important notion is the dual of morphisms. This is done diagrammatically, as follows. Denote a morphisms  $f: A \rightarrow B$  as in equation (??). The dual morphism  $f^*$  is

then defined by the diagram

$$\begin{array}{ccc} A^* & & A^* \\ | & & | \\ \boxed{f^*} & = & \boxed{f} \\ | & & | \\ B^* & & B^* \end{array} \quad (B.36)$$

## B.5 Braiding

The braiding in a category is governed by the definition of a commutativity constraint. This is a natural isomorphism from the tensor product functor  $\otimes$  to the opposite tensor product  $\otimes^{\text{op}}$ , which is defined through

$$A \otimes^{\text{op}} B = B \otimes A \quad (B.37)$$

This means that for any objects  $A$  and  $B$ , there is an isomorphism

$$c_{A,B}: A \otimes B \rightarrow B \otimes A \quad (B.38)$$

Since it is an isomorphism, there is an inverse  $c_{A,B}^{-1}: B \otimes A \rightarrow A \otimes B$ , but we do not necessarily have  $c_{A,B}^{-1} = c_{B,A}$  (since that would imply that the monodromy would be trivial  $c_{A,B}c_{B,A} = \text{id}_{A \otimes B}$ ), which gives the interesting braiding properties corresponding to anyonic excitations.

The associativity constraint must satisfy the *Hexagon axiom*. Is formulated as the statements that the diagrams

$$\begin{array}{ccc} A \otimes (B \otimes C) & \xrightarrow{c_{A,B \otimes C}} & (B \otimes C) \otimes A \\ \alpha_{A,B,C} \nearrow & & \searrow \alpha_{B,C,A} \\ (A \otimes B) \otimes C & & B \otimes (C \otimes A) \\ c_{A,B} \otimes \text{id}_C \searrow & & \nearrow \text{id}_B \otimes c_{A,C} \\ (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) \end{array} \quad (B.39)$$

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and commute

$$\begin{array}{ccccc}
 & & A \otimes (B \otimes C) & \xrightarrow{c_{A,B \otimes C}^{-1}} & (B \otimes C) \otimes A \\
 & \nearrow^{\alpha_{A,B,C}} & & & \searrow^{\alpha_{B,C,A}} \\
 (A \otimes B) \otimes C & & & & & B \otimes (C \otimes A) \\
 & \searrow_{c_{A,B}^{-1} \otimes \text{id}_C} & & & \nearrow_{\text{id}_B \otimes c_{A,C}^{-1}} \\
 & & (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C)
 \end{array} \tag{B.40}$$

(the diagrams differ by  $c \rightarrow c^{-1}$ ).

The braiding is of course expressed diagrammatically by ‘over’ and ‘under crossings’.

So

$$c_{A,B} = \begin{array}{c} \diagdown \\ A \diagup B \end{array}, \quad c_{A,B}^{-1} = \begin{array}{c} \diagup \\ A \diagdown B \end{array} \tag{B.41}$$

Which elucidates the correspondence to the pentagon relations from chapter 4.

Where the Pentagon axiom ensures that arbitrary diagrams of composition of the associativity constraints commute, the Hexagon axiom makes sure that any composition of associativity constraints and braidings (or inverse braidings) commutes. The first means that any two tensor products tensor products  $(A_1 \otimes \dots \otimes A_k)_B$  and  $(A_1 \otimes \dots \otimes A_k)_{B'}$ , with different brackets  $B$  and  $B'$ , are canonically isomorphic. The second implies that tensor products of the same objects put in arbitrary order are also canonically isomorphic.

### B.5.1 Ribbon structure

The non-trivial twist of anyons is given by the so called ribbon structure. This consists of a natural isomorphism  $\theta_A$  such that  $\theta_A: A \rightarrow A$  is an isomorphism, with

$$\theta_{A^*} = \theta_A^* \tag{B.42}$$

$$\theta_{\mathbf{1}} = \text{id}_{\mathbf{1}} \tag{B.43}$$

$$\tag{B.44}$$

and the following diagram commutes

$$\begin{array}{ccc}
 A \otimes B & \xrightarrow{c_{A,B}} & B \otimes A \\
 \theta_{A \otimes B} \downarrow & & \downarrow \theta_{B \otimes A} \\
 A \otimes B & \xleftarrow{c_{B \otimes A}} & B \otimes A
 \end{array} \tag{B.45}$$

When writing out this equation in terms of simple objects, one essentially finds the monodromy equation.



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## APPENDIX C

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### Data for $\mathfrak{su}(2)_{10}$

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Below are the matrices  $S$  and  $\bar{S}$  for  $\mathfrak{su}(2)_{10}$  defined through the diagrammatic equations

$$S_{ab} = \text{Diagram 1}, \quad \bar{S}_{ab} = \text{Diagram 2} \quad (\text{C.1})$$

The diagrammatic equation (C.1) defines the matrices  $S$  and  $\bar{S}$ . The left side shows  $S_{ab}$  as two overlapping circles with arrows forming a cycle between indices  $a$  and  $b$ . The right side shows  $\bar{S}_{ab}$  as a vertical capsule shape with two internal loops and arrows, with indices  $a$ ,  $b$ , and  $6$  labeled.

The indices  $a, b$  run over the eleven charge labels  $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$  of this theory.



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