A micro-course in

Spectral Sub-Riemannian Geometry

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DDT & G

PART ONE

An Introduction to sub-Riemannian geometry

The many faces of Sub-Riemannian Geometry

Carnot-Charatéodory geometry
(GROMOV, 1981)

Singular Riemannian geometry
(BROCKETT, 1981)

Sub-Riemannian geometry
(STRICHARTZ, 1986)*

Non-holonomic geometry
(FROM MECHANICS)

Sub-elliptic geometry
(FROM PDEs)

And possibly even more ...

* Strichartz, Sub-Riemannian Geometry, J. Diff. Geom. 24(2) 1986 + ERRATA 30(2) 1989

Recommended Readings (PDFs available from authors' webpages in most cases!)

- ☆ R. Montgomery. A tour of Sub-Riemannian geometries, their geodesics and applications. AMS (2002)
- ☆ L. Rifford. Sub-Riemannian Geometry & Optimal transport. SpringerBriefs (2014)
- ☆ F. Jean. Control of non-holonomic systems: from sub-Riemannian geometry to motion planning. SpringerBriefs (2014)
- ☆ E. Le Donne. Lecture notes on sub-Riemannian geometry. Unpublished, PDF available at enrico.ledonne.googlepages.com (2017)
- ☆ A. Agrachev, D. Barilari, U. Boscain. A comprehensive introduction to sub-Riemannian geometry. CUP (2019)
- ☆ Video courses from CIRM Summer School & IHP Trimester (2014)
  - www.cmap.polytechnique.fr/subriemannian/cirm
  - www.cmap.polytechnique.fr/~subriemannian

Outline of Part I

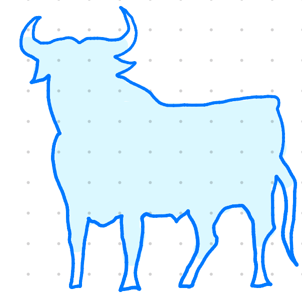
- From the fundation of Carthago to sub-Riemannian geodesics.
  A prototypical example

- Sub-Riemannian structures 101

- Geometry meets Optimal Control.
  Into the Dungeon of sub-Riemannian geodesics

Mythological origins: Dido problem

Legend* says that Dido arrived in 814 BC on the coast of Tunisia and asked for a piece of land to buy. The king of North Africa granted her request with as much coastline land as she could enclose with a bull's hide.



* Virgil, The Aeneid

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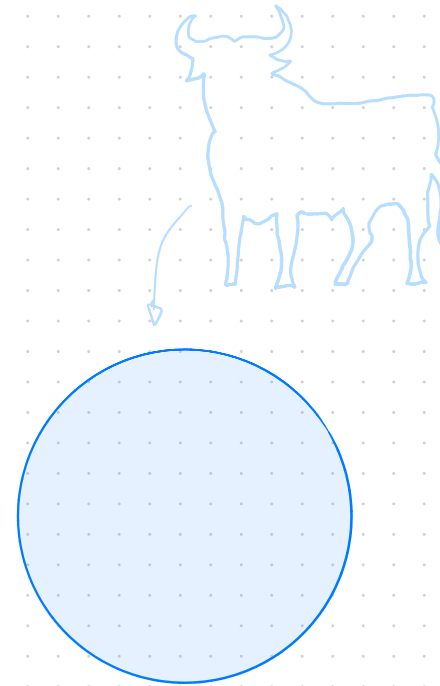

Dido cut the hide into a long thin strip and used it to encircle the land.

$\boxed{\text{This became Chartage and Dido its Queen!}}$

But a true mathematical solution had to wait the 19th century!

* Virgil, The Aeneid

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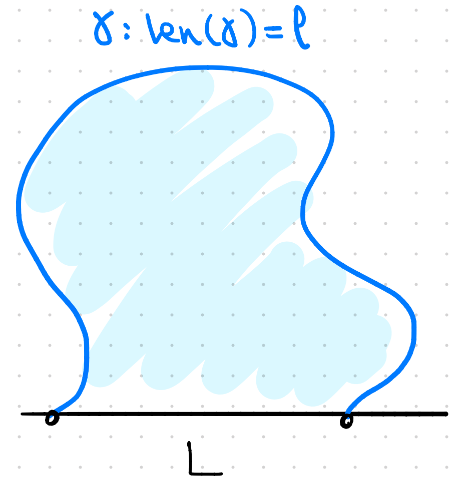


L: Mediterranean coastline

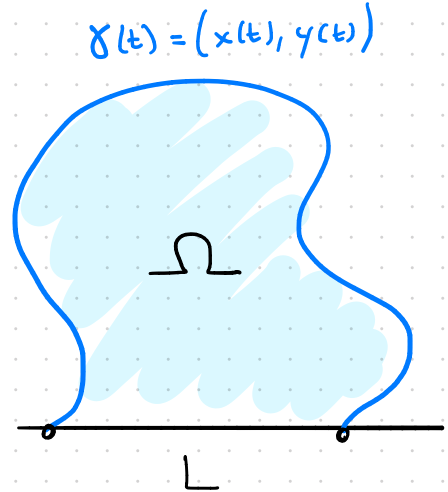
ℓ: bull's hide (a string of fixed lenght)

FIX ℓ AT THE ENDPOINTS OF L, THEN MOVE THE ROPE. YOU WANT TO FIND THE SHAPE FOR THE CURVE γ THAT ENCLOSES THE LARGEST AREA OVER L.

Mythological origins: Dido problem

Fix the area $A(\Omega)$ of Ω.

Find shortest $\gamma(t)$ enclosing a region with the fixed area.



Recall:

$$\mathrm{len}(\gamma) = \int \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, dt$$ ← WE WOULD LIKE TO MINIMIZE THIS

$$A(\Omega) = \int_\Omega dx \wedge dy$$ ← KEEPING THIS FIXED

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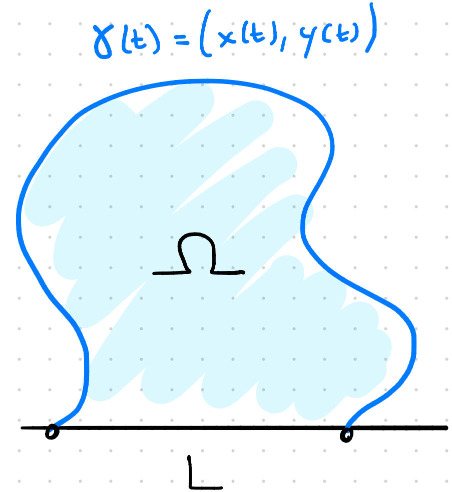

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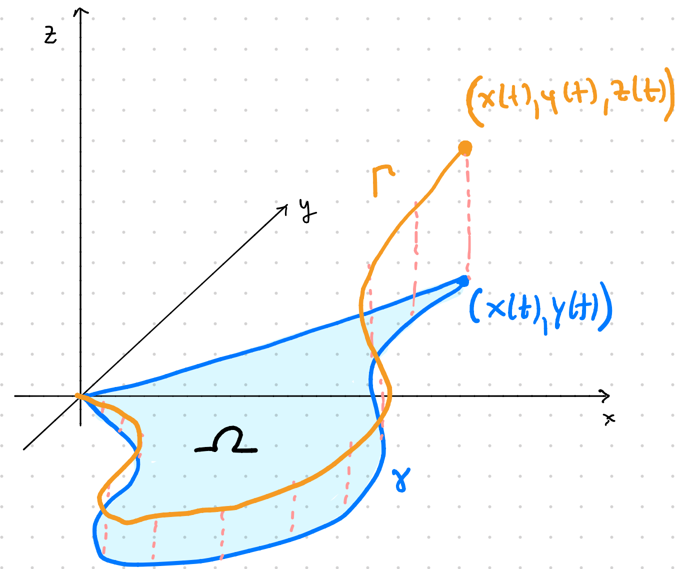
$$= \int_\Omega \frac{1}{2} d(x\,dy - y\,dx) = \int_\gamma \frac{1}{2}(x\,dy - y\,dx)$$

Dido meets Analysis on Manifolds

lift $\gamma \subset \mathbb{R}^2$ to $\Gamma \subset \mathbb{R}^3$ by

$$\gamma : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \Gamma : \begin{pmatrix} x \\ y \\ z = A(\Omega) \end{pmatrix}$$



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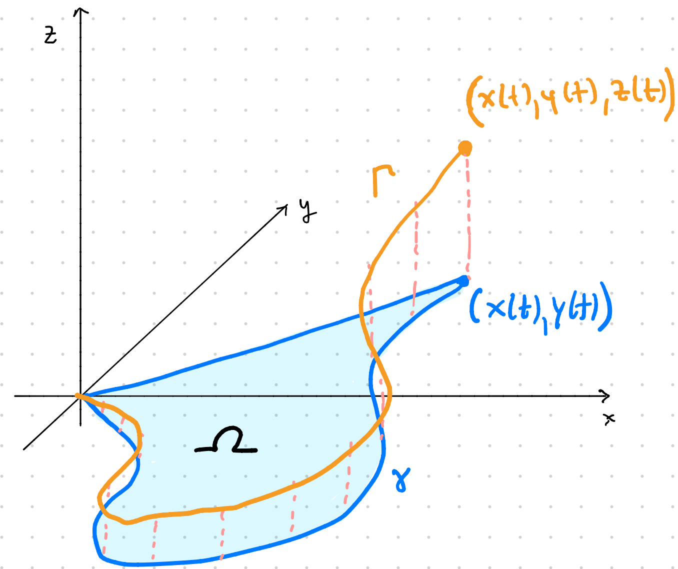

Since $\quad z := A(\Omega) = \int_\gamma \frac{1}{2}(x\,dy - y\,dx)$

Γ satisfies a non-holonomic constraint

$$\dot{z}(t) = \frac{1}{2}\left(x(t)\,\dot{y}(t) - y(t)\,\dot{x}(t) \right)$$

Dido meets Analysis on Manifolds (non-holonomic?)

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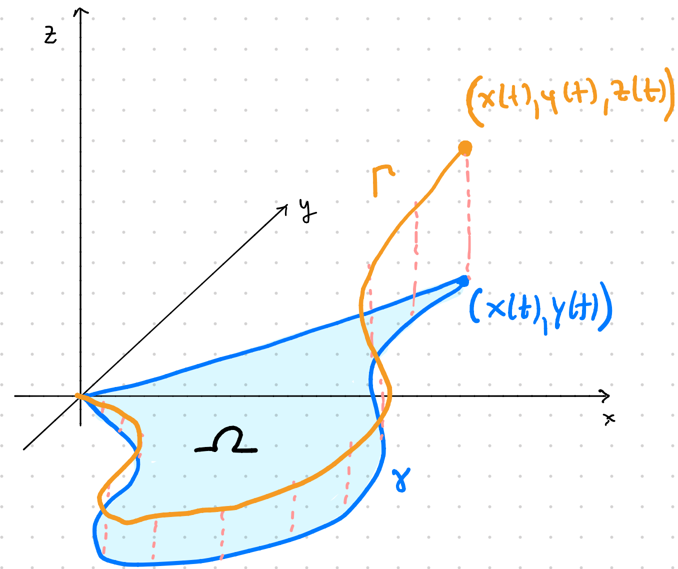

$\Rightarrow$ Γ satisfies the non-holonomic constraint

$$\omega(\dot{\Gamma}) = 0$$

$$\omega := dz - \frac{1}{2}(x\,dy - y\,dx)$$

Dido meets Analysis on Manifolds (non-holonomic!)

lift $\gamma \subset \mathbb{R}^2$ to $\Gamma \subset \mathbb{R}^3$ by

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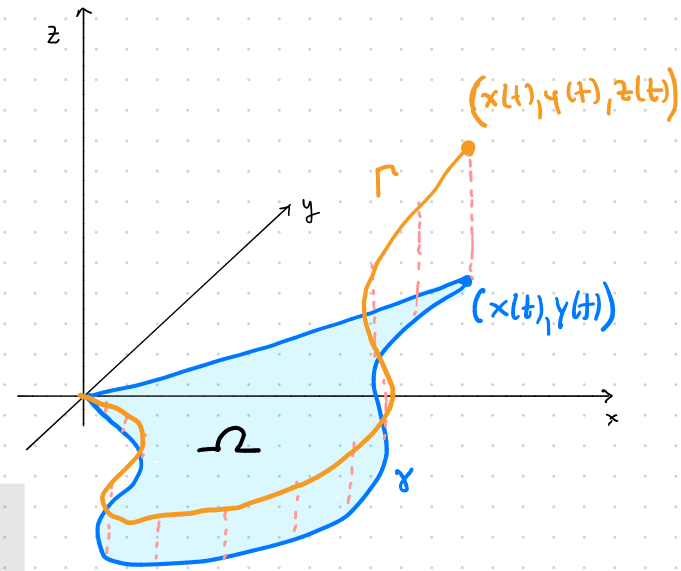


$\Rightarrow$ Γ satisfies the non-holonomic constraint

$\omega(\dot{\Gamma}) = 0$ — the subspace of ADMISSIBLE VELOCITIES is $\ker(\omega)$!

$$\omega := dz - \frac{1}{2}(x\,dy - y\,dx)$$

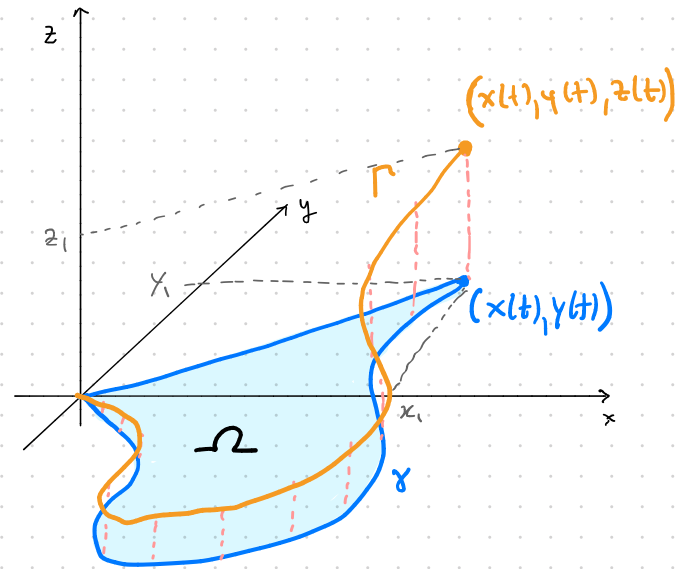
That is, there is a constraint on what are admissible directions of movement

Dido meets Analysis on Manifolds

lift $\gamma \subset \mathbb{R}^2$ to $\Gamma \subset \mathbb{R}^3$ by $z = A(\Omega)$

$\Rightarrow$ Γ satisfies the non-holonomic constraint

$$\omega(\dot{\Gamma}) = 0 \qquad \omega := dz - \frac{1}{2}(x\,dy - y\,dx)$$



$\Rightarrow$ Dido problem, alternative formulation

Find $\Gamma : [0,1] \longrightarrow \mathbb{R}^3$ such that

- $\omega(\dot{\Gamma}) = 0$,
- $\Gamma(0) = (0,0,0)$, $\Gamma(1) = (x_1, y_1, z_1)$
- len(γ is minimal)

Dido meets Analysis on Manifolds

lift $\gamma \subset \mathbb{R}^2$ to $\Gamma \subset \mathbb{R}^3$ by $z = A(\Omega)$

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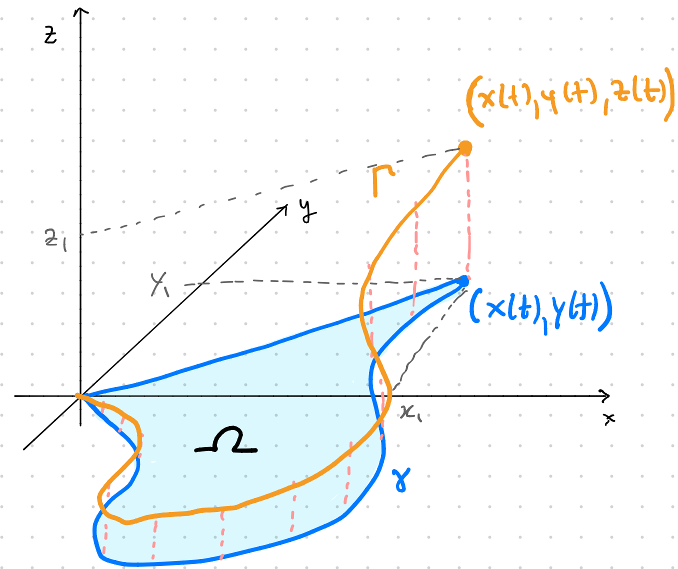

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The fixed area $A(\Omega)$

The end of the coastline segment

Dido meets Analysis on Manifolds

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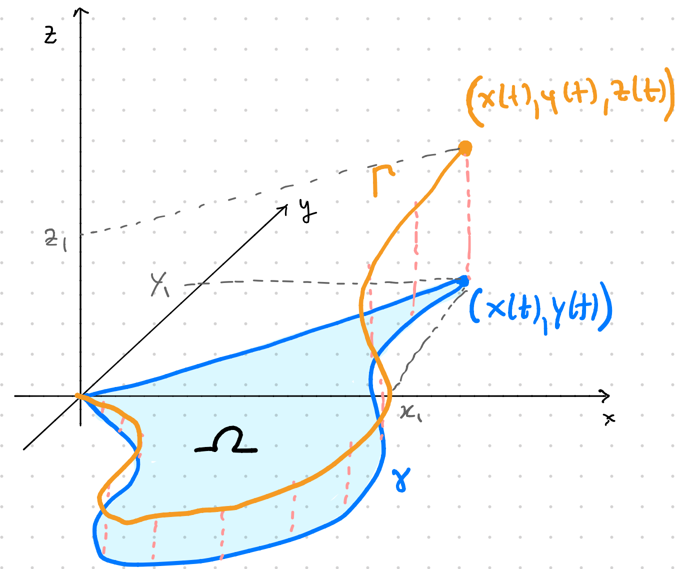

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- $\text{len}(\gamma)$ is minimal

In other words

$$\text{len}(\Gamma) = \int_0^1 \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, dt$$

Dido meets Analysis on Manifolds

lift $\gamma \subset \mathbb{R}^2$ to $\Gamma \subset \mathbb{R}^3$ by $z = A(\Omega)$

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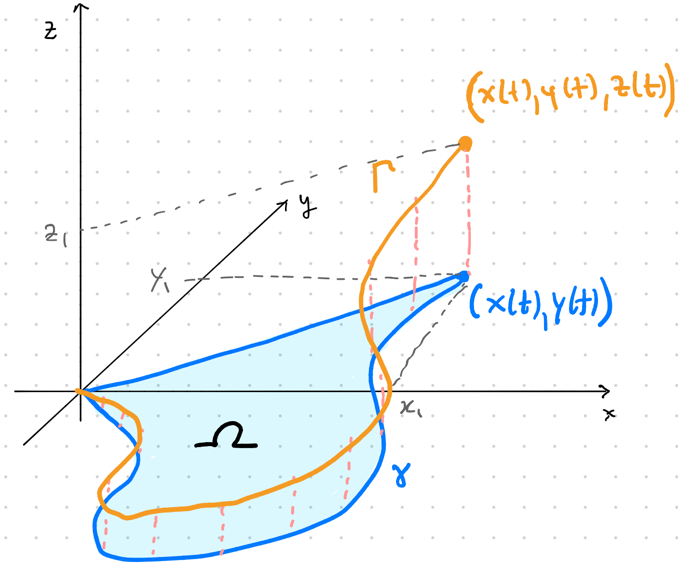

$\Rightarrow$ Dido problem, alternative formulation

Ambient space

Find $\Gamma : [0,1] \longrightarrow \mathbb{R}^3$ such that

Model of Sub-Riemannian manifold

- $\omega(\dot\Gamma) = 0$, Constraints on $T\mathbb{R}^3$
- $\Gamma(0) = (0,0,0)$, $\Gamma(1) = (x_1, y_1, z_1)$
- $\ell en(\Gamma) = \int_0^1 \sqrt{\dot x^2(t) + \dot y^2(t)}\, dt$ is minimal

A notion of length that plays nice with the constraints

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Sub-Riemannian manifolds, at last

A **sub-Riemannian manifold** is a triple (M, D, g)

- M is a smooth manifold
- $D \subset TM$ is a vector distribution
- g is a Riemannian metric on D

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Dido's Problem

$M = \mathbb{R}^3$

$D = \ker(\omega) \subset T\mathbb{R}^3$

$g = dx^2 + dy^2$

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HORIZONTAL ~~ADMISSIBLE~~ **CURVE**: $\gamma \in [0,1] \longrightarrow M$ such that $\dot{\gamma} \in D_{\gamma(t)}$

These are the "lifts" Γ 🙇

$$\Rightarrow \; \mathrm{len}_g(\gamma) := \int_0^1 \|\dot{\gamma}(t)\|_g \, dt$$

$$\int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2} \, dt$$

Sub-Riemannian manifolds, at last

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DIDO'S PROBLEM

$M = \mathbb{R}^3$

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In the sub Riemannian world it is common to work in terms of a local basis of vector fields for the distribution:

$$D = \operatorname{span}\{X_1, \dots, X_k\} \subset TM$$

$k = k(q)$, THE DIMENSION OF THE BASIS IS THE RANK OF THE DISTRIBUTION.

$$\forall q \in M \quad D_q = \operatorname{span}\{X_1|_q, \dots, X_k|_q\} \subset TM$$

Sub-Riemannian manifolds

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- M is a smooth manifold
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Dido's Problem

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$D = \ker(\omega) \subset T\mathbb{R}^3$

$g = dx^2 + dy^2$

$$D = \mathrm{Span}\{X_1, \dots, X_k\} \subset TM$$

$$X_1 = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}$$

$$X_2 = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}$$

Sub-Riemannian manifolds

A **sub-Riemannian manifold** is a triple (.

- M is a smooth manifold
- $D \subset TM$ is a vector distr
- g is a Riemannian

$D = \text{Span}$

HOW DO WE KNOW THAT THIS MAKES ANY SENSE?

DO THE MINIMIZERS "Γ" REALLY EXIST?

DIDO'S PROBLEM

$M = \mathbb{R}^3$

$= \ker(\omega) \subset T\mathbb{R}^3$

$dx^2 + dy^2$

$X_1 = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}$

$X_2 = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}$

Existence of admissible lenght minimizing curves

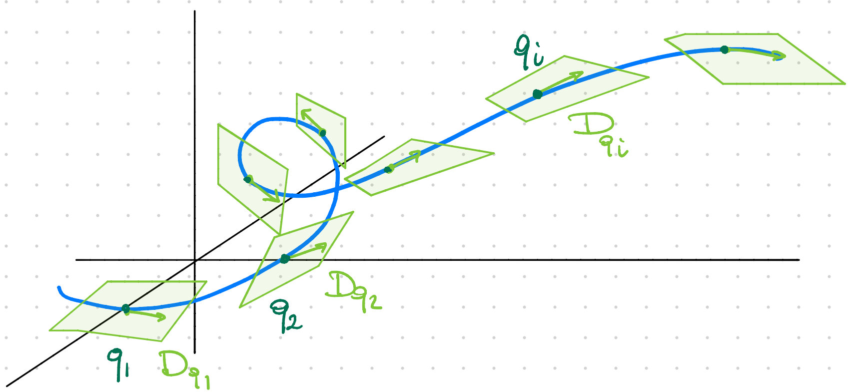
$$X_1 = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}$$

$$X_2 = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}$$

$$\Rightarrow \quad [X_1, X_2] = \frac{\partial}{\partial z} \notin D$$

$$\left[X_1, \frac{\partial}{\partial z}\right] = \left[X_2, \frac{\partial}{\partial z}\right] = 0$$

Back to Dido, how can we see if our geometric construction makes sense?



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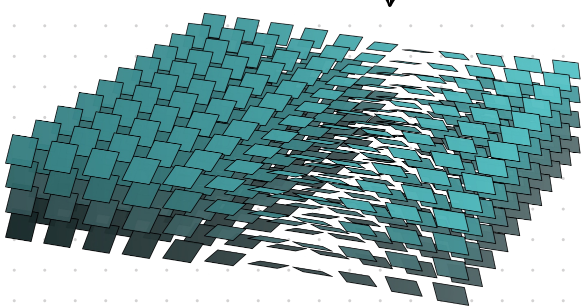
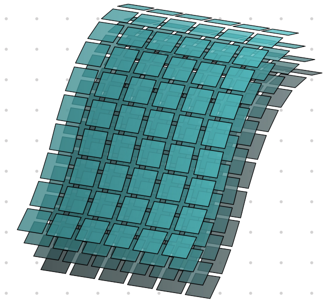
$$D = \mathrm{Span}\{X_1, X_2\}$$

$$\mathrm{Span}\{X_1, X_2, [X_1, X_2]\} = TM$$



! This is for a different basis but helps to get the idea

This is crucial!
If $[D, D] \subseteq D$, by Frobenius thm M is foliated by integral submanifolds.



Figures: Wikimedia fundation

Existence of admissible lenght minimizing curves

Hörmander condition

$\exists s \in \mathbb{N}$ such that

$$\operatorname{span}\left\{ [X_{i_1},[X_{i_2},\dots,[X_{i_{m-1}},X_{i_m}]\dots]]_q \;\middle|\; 1\le m\le s \right\} = T_qM \qquad \forall q\in M$$

$\Rightarrow$ $\mathcal{D}$ is BRACKET GENERATING (or COMPLETELY NON-INTEGRABLE / NON-HOLONOMIC)

$s = s(q)$ is STEP of distribution

Existence of admissible lenght minimizing curves

<u>Hörmander condition</u>

$$\text{span}\{[X_{i_1},[X_{i_2},\dots,[X_{i_{m-1}},X_{i_m}]\dots]]_q\} = T_qM \quad \forall q\in M$$

$D = \text{span}\{X_1,\dots,X_k\}$

FROM NOW ON IS ASSUMED TO BE BRACKET GENERATING

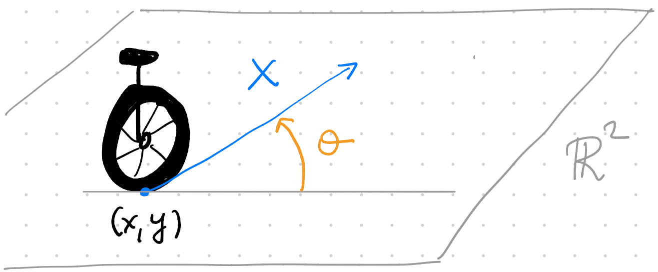
'39 '38

Chow - Rashevski

D bracket generating $\Rightarrow$ $\exists\, U_q\subset M$, $U_q\ni q$, such that

$\forall x,y\in U_q$ $\exists\,\gamma$ admissible such that $\gamma(0)=x$, $\gamma(1)=y$

Examples (Unicycle)



<u>Three variables</u>

- $(x,y) \in \mathbb{R}^2$ position
- θ orientation of the wheel

$\Rightarrow$ $M = \mathbb{R}^2 \times S^1$

<u>Possible motions</u>

- FORWARD (BACKWARD) MOTION

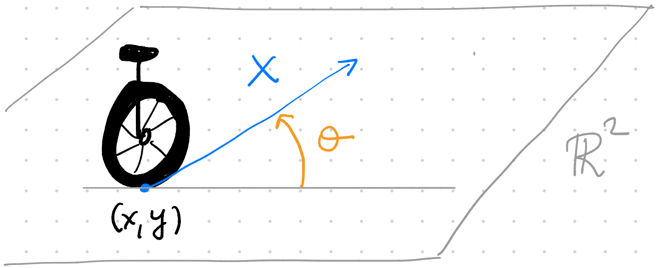
$$X := \cos\theta \, \partial_x + \sin\theta \, \partial_y$$

- STEERING IN PLACE

$$Y := \partial_\theta$$

$$\mathcal{D} = \operatorname{span}\{X, Y\}$$

Examples (Unicycle)



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- STEERING IN PLACE

$$Y := \partial_\theta$$

$\mathcal{D} = \operatorname{span}\{X, Y\} \subset TM$

HOWEVER

$$[X, Y] = -\sin\theta\, \partial_x + \cos\theta\, \partial_y =: Z$$

$\Rightarrow$ "ORTHOGONAL" MOTION POSSIBLE

AND $\operatorname{span}\{X, Y, [X,Y]\} = TM$!

Existence of admissible lenght minimizing curves

Chow - Rashevski

$\mathcal{D}$ bracket generating $\Rightarrow$ $\exists\, U_q \subset M$, $U_q \ni q$, such that

$\forall\, x, y \in U_q \quad \exists\, \gamma$ admissible such that $\gamma(0) = x$, $\gamma(1) = y$

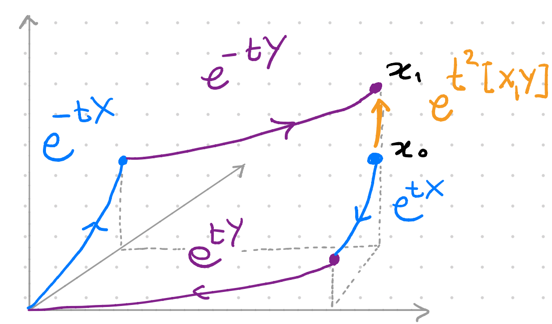
Key idea in proof

You can achieve "non-horizontal directions" by means of commutators

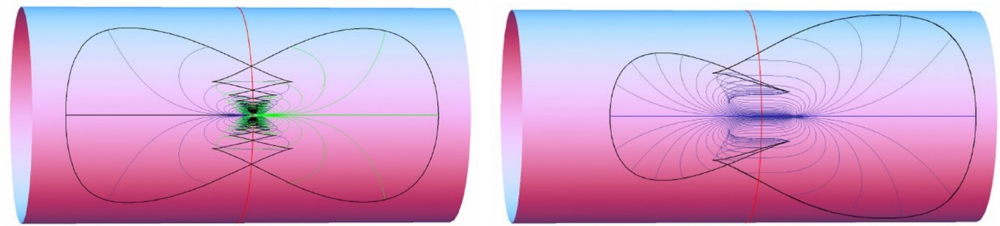
$$e^{-tY} e^{-tX} e^{tY} e^{tX} = \mathbb{1} + e^{t^2 [X,Y]} + o(t^2)$$

but you go slower

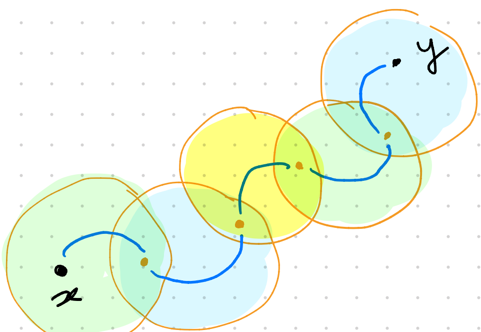
BRACKET GENERATING CRUCIAL TO BE ABLE TO ACHIEVE ANY DIRECTION



Sub-Riemannian metric



$$d_{SR}(x,y) := \inf \{ \operatorname{len}(\gamma) \mid \gamma \text{ admissible}, \ \gamma(0)=x, \ \gamma(1)=y \}$$



Thanks to Chow-Rashevski's theorem, we know these exist!

Figures: [Boscain, Prandi, Seri]

Sub-Riemannian metric (or Carnot-Carathéodory distance)

$$d_{SR}(x,y) := \inf \{ \operatorname{len}(\gamma) \mid \gamma \text{ admissible}, \ \gamma(0)=x, \ \gamma(1)=y \}$$

By Chow-Rashevski one can also show

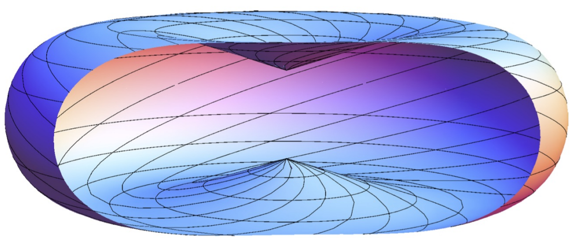
- (M, d_{SR}) has the same topology as M
- $d_{SR} : M \times M \to \mathbb{R}$ is finite and continuous

$\Rightarrow$ We found a well-defined new metric!

Examples (~~Dido~~)

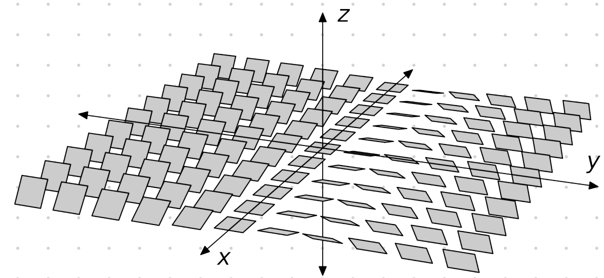
Heisenberg group $\mathbb{H}_3$

$M = \mathbb{R}^3$ $\qquad \mathcal{D} = \mathrm{span}\left\{\partial_x - \frac{y}{2}\partial_z,\ \partial_y + \frac{x}{2}\partial_z\right\} = \mathrm{span}\left\{\partial_y\ ,\ \partial_x + y\,\partial_z\right\}$



Unit sphere for $\mathbb{H}^3$

[Agrachev, Barilari, Boscain]



Heisenberg distribution

[Wikimedia foundation]

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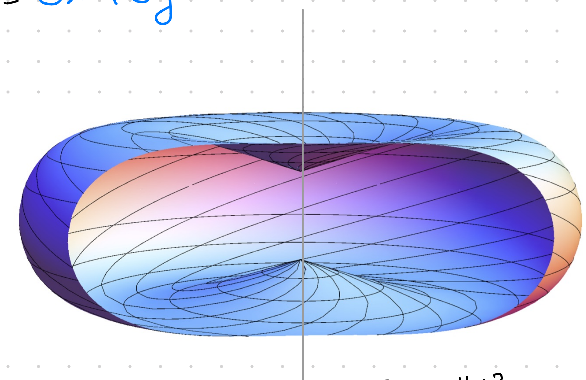


Unit sphere for $\mathbb{H}^3$

[Agrachev, Barilari, Boscain]

d_{SR} never smooth nor lipschitz if $x=y$!

Flags and dimensions (A brief technical parenthesis)

Canonical flag of a distribution D of step s is filtration

$$\{0\} = D^0_q \subset \underset{\substack{\shortparallel \\ D_q}}{D^1_q} \subset \cdots \subset D^j_q \subset \cdots \subset D^{s(q)}_q = T_qM$$

$D^j_q = D^{j-1}_q + [D_q, D^{j-1}_q]$
span of applications of j Lie brackets

$\Rightarrow$ Growth vector $k^D(q) = (\dim D^1_q, \ldots, \dim D^{s(q)}_q)$

If $k^D(q) \equiv k^D$ constant $\Rightarrow$ D is called EQUIREGULAR

DIDO: $D^0 = \{0\}$, $D^1 = \mathrm{Span}\{\partial_x, \partial_y + x\partial_z\}$, $D^2 = TM$

$k^D(q) = (2, 3)$ CONSTANT

Flags and dimensions (more buzzwords for experts)

FILTRATION $\quad \{0\} = D_q^0 \subset D_q^1 \subset \cdots \subset D_q^j \subset \cdots \subset D_q^{s(q)} = T_qM$

$D_q^1 = D_q$

NO GRADATION but well defined graded vector space (nilpotentization of D)

$$gr_q(D) = D_q^1 \oplus D_q^2 / D_q^1 \oplus \cdots \oplus D_q^s / D_q^{s-1}$$

$\leftarrow$ D EQUIREGULAR $\Rightarrow$ Homogeneous Stratified Lie algebra whose Lie group is called Carnot group

DIDO: $D^1 = \mathrm{span}\{\partial_x, \partial_y + x\partial_z\}$,

$D^2 / D^1 = \mathrm{span}\{\partial_z\}$

$\Rightarrow \mathbb{H}^3$ is left-invariant sub-Riemannian structure on $G = \mathbb{R}^3$ with product

$$(x, y, z) \odot (x', y', z') := \left(x + x', y + y', z + z' + \tfrac{1}{2}(xy' - x'y)\right)$$

Step 2 Carnot group

Flags and dimensions

FILTRATION $\{0\} = D_q^0 \subset \underset{\underset{D_q}{\shortparallel}}{D_q^1} \subset \cdots \subset D_q^j \subset \cdots \subset D_q^{s(q)} = T_qM$

NO GRADATION but well defined graded vector space

$$gr_q(D) = D_q^1 \oplus D_q^2 / D_q^1 \oplus \cdots \oplus D_q^m / D_q^{m-1}$$

$\Rightarrow$ Privileged (canonical) coordinates

- if D equiregular $\Rightarrow$ smooth system of privileged coordinates
- in general existence of continuous system of coordinates around a singular point is not guaranteed

Examples

① Contact structures

M dim $2n+1$

$\mathcal{D} = \ker(dz - y\,dx)$

$= \mathrm{span}\{\partial y_i,\ \partial x_i + y_i \partial z\}$

STEP 2 EQUIREGULAR

② Quasi-contact structures

M dim $2n+2$

$\mathcal{D} = \mathrm{span}\{\partial y_i,\ \partial x_i + y_i \partial z,\ \partial w\}$

STEP 2 EQUIREGULAR

③ Grushin plane

M (dim 2)

$\mathcal{D} = \mathrm{span}\{\partial x,\ x\partial y\}$

STEP 2 NOT EQUIREGULAR

④ Engel structure

M dim 4

$\mathcal{D} = \mathrm{span}\{\partial x,\ \partial y + x\partial z + z\partial w\}$

STEP 3 EQUIREGULAR

⑤ Martinet structure

M dim 3

$\mathcal{D} = \ker(dz - y^2 dx)$

$= \mathrm{span}\{\partial y,\ \partial x + \frac{y^2}{2}\partial z\}$

STEP 3 EQUIREGULAR

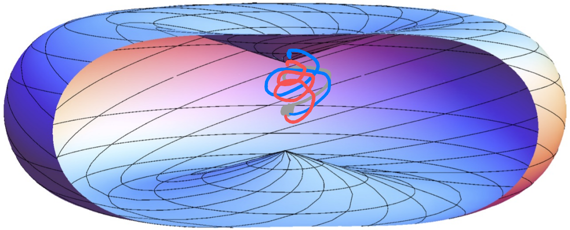
⑥ (???)

M dim 3

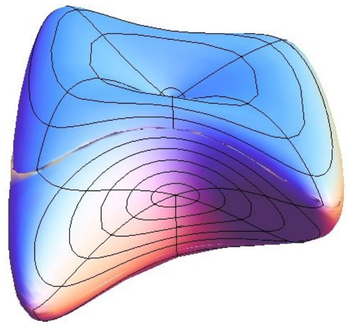
$\mathcal{D} = \mathrm{span}\{\partial x,\ \partial y + xz\,\partial z\}$

$z=0 \Rightarrow$ NOT BRACKET GENERATING

Topology of balls

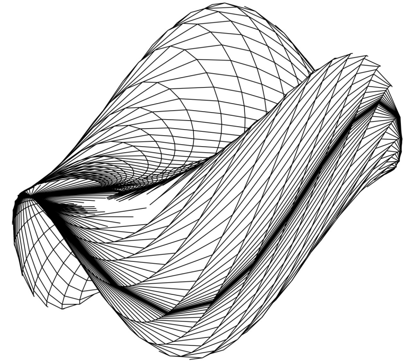


HEISENBERG BALL



BALL ON THE SR PROBLEM ON THE LIE GROUP SH(2)

[Butt et al. 2014]



MARTINET BALL

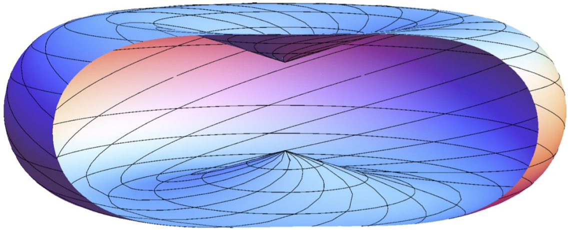
[Agrachev et. al. 1997]

Sub-Riemannian balls are clearly weird!

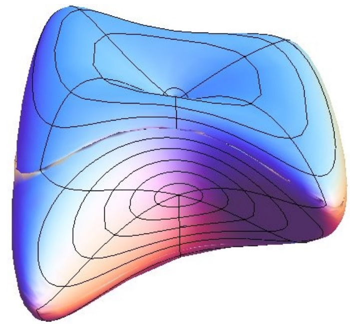
$$B_c(r) = \{x \in M \mid d_{SR}(x,c) < r\}$$

$$S_c(r) := \partial_{top} B_c(r)$$

Topology of balls (OPEN PROBLEM)



HEISENBERG BALL



BALL ON THE SR PROBLEM ON THE LIE GROUP SH(2)

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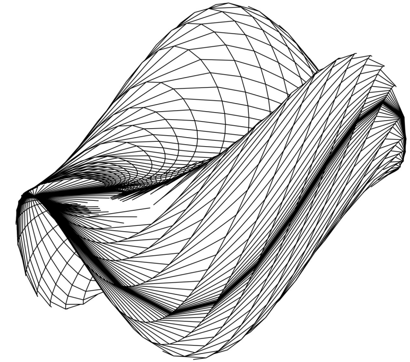

MARTINET BALL

[Agrachev et. al. 1997]

Q. ARE SMALL SR BALLS HOMEOMORPHIC TO EUCLIDEAN BALLS?

A. We don't know in general.

[Baryshnikov, GAFA, 2000] Yes for step 2 and for Carnot groups

Outline of Part I

- From the fundation of Cartkaclo to sub-Riemannian geodesics.
  A prototypical example

- Sub-Riemannian structures 101

- Geometry meets Optimal Control.
  Into the Dungeon of sub-Riemannian geodesics

Sub-Riemannian Geodesics

Let's fix a sub-Riemannian manifold $(M, \mathcal{D}, g)$. Let $x, y \in M$.

A GEODESIC is an admissible curve $\gamma : [0,1] \longrightarrow M$, with $\gamma(0) = x$ and $\gamma(1) = y$, which minimises the length functional

$$\operatorname{len}_g(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_g \, dt$$

Equivalently, γ minimises the energy functional $J_g(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|^2 dt$
(modulo time reparametrization)

Sub-Riemannian Geodesics

Let's fix a sub-Riemannian manifold (M, D, g). Let $x, y \in M$.

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$$\mathrm{len}_g(\gamma) = \int_0^1 \|\dot\gamma(t)\|_g \, dt \qquad \text{equiv} \qquad J_g(\gamma) = \int_0^1 \|\dot\gamma(t)\|^2 dt$$

Riemannian geometry

- $\nabla_{\dot\gamma} \dot\gamma = 0$
- Hamiltonian formulation: $H(p,q) = \frac{1}{2} g_q^{\alpha\beta} p_\alpha p_\beta$
- Euler-Lagrange equations $\rightsquigarrow$ Unique parametrization by initial speed

Sub-Riemannian Geodesics

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Riemannian geometry

Non holonomic nature of the problem breaks this down!

- $\nabla_{\dot\gamma} \dot\gamma = 0$
- Hamiltonian formulation: $H(p,q) = \frac{1}{2} g_q^{\alpha\beta} p_\alpha p_\beta$
- Euler-Lagrange equations $\rightsquigarrow$ Unique parametrization by initial speed

Sub-Riemannian Geodesics - Optimal control to the rescue

ASSUMPTION $\mathcal{D} = \text{span}\{X_1, \dots, X_k\}$ GLOBAL orthonormal frame

A curve γ is HORIZONTAL (ADMISSIBLE) if there exist $\underbrace{u \in L^2([0,1];\mathbb{R}^k)}_{\text{CONTROL}}$ solving the Cauchy problem

$$\dot{\gamma}(t) = \sum_{i=1}^{k} u_i(t) X_i(\gamma(t))$$

Sub-Riemannian Geodesics - Optimal control to the rescue

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Sub-Riemannian Geodesics - Optimal control to the rescue

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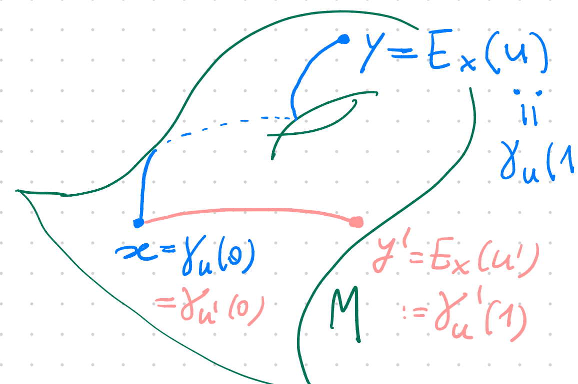
$$\dot\gamma(t) = \sum_{i=1}^{k} u_i(t) X_i(\gamma(t)) \qquad \gamma(0) = x$$

$\leftarrow$ $\boxed{\text{WE DENOTE THE SOLUTION } \gamma_u}$

$\Rightarrow$ END-POINT MAP

$$E_x : L^2([0,1];\mathbb{R}^k) \to M$$
$$u \longmapsto \gamma_u(1)$$

is SMOOTH and OPEN.



Sub-Riemannian Geodesics - Optimal control to the rescue

locally always possible

ASSUMPTION $D = \text{span}\{X_1, \dots, X_k\}$ GLOBAL orthonormal frame

Horizontal curve γ_u solves Cauchy problem

$$\dot{\gamma}_u(t) = \sum_{i=1}^{k} u_i(t) X_i(\gamma_u(t)), \quad \gamma_u(0) = x, \quad u \in L^2([0,1]; \mathbb{R}^k)$$

End-point map $E_x(u) := \gamma_u(1)$

$\Rightarrow$ GEODESIC PROBLEM $\min J(\gamma) = \min \int_0^1 \|\dot{\gamma}\|_g^2 \, dt, \quad \begin{array}{l} \gamma(0) = x \\ \gamma(1) = y \end{array}$

Sub-Riemannian Geodesics - Optimal control to the rescue

locally always possible

ASSUMPTION $D = \mathrm{span}\{X_1, \dots, X_k\}$ GLOBAL orthonormal frame

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$$= \min \int_0^1 \Big\| \sum u_i^\gamma X_i(\gamma) \Big\|_g^2 dt = \min \int_0^1 \|u^\gamma(t)\|^2 dt, \quad E_x(u^\gamma) = y$$

$$=: \min J(u^\gamma)$$

CONSTRAINED MINIMUM PROBLEM

Sub-Riemannian Geodesics - Lagrange multipliers

GOAL find minimizers of $J(u) := \int_0^1 \|u(t)\|^2 dt$, $u \in E_x^{-1}(y)$

IDEA use Lagrange multipliers (LM)

PROBLEM LM is fine if y is REGULAR VALUE of E_x, that is, if $D_u E_x$ SURJECTIVE

Sub-Riemannian Geodesics - Lagrange multipliers

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NECESSARY CONDITION FOR MINIMALITY

If γ_u minimizer for J
$\Rightarrow \exists \lambda \in T_y^*M$, $\chi \in \{0,1\}$ s.t.

$$\chi D_u J - \lambda D_u E_x = 0$$

Sub-Riemannian Geodesics - Lagrange multipliers

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NECESSARY CONDITION FOR MINIMALITY

If γ_u minimizer for J
$\Rightarrow \exists \lambda \in T_y^* M$, $\chi \in \{0,1\}$ s.t.

$$\chi D_u J - \lambda \cdot D_u E_x = 0$$

$\chi = 1$: NORMAL CASE

$\chi = 0$: $\lambda \cdot D_u E_x = 0$ ABNORMAL CASE

$\lambda \neq 0$

(λ, χ) NOT UNIQUE: a minimiser can be BOTH normal AND abnormal

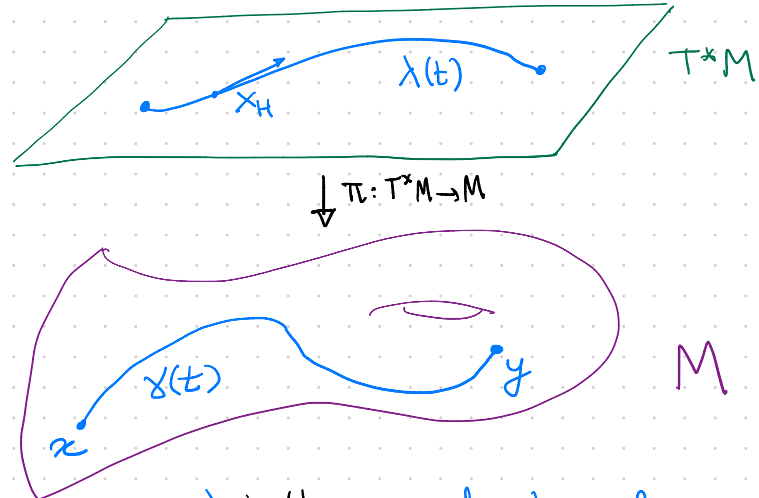
Normal ~~minimizers~~ extremals

Thanks to Lagrange Multipliers, a normal extremal is the projection $\pi(\lambda)$ of a solution $\lambda(t)$ of the

SUB-RIEMANNIAN HAMILTONIAN

$$H : T^*M \longrightarrow \mathbb{R}$$

$$H(p,x) := \frac{1}{2}\sum_{i=1}^{k} (p \cdot X_i(x))^2$$



λ is the normal extremal lift of γ in T^*M

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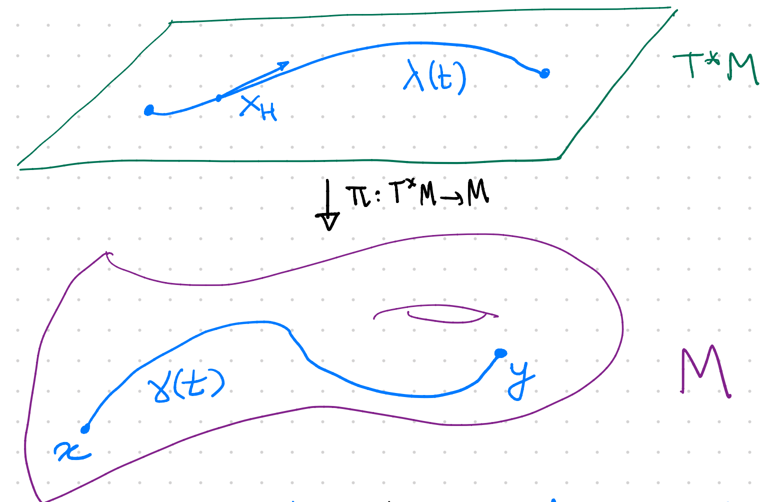


λ is the normal extremal lift of γ in T^*M

Standard Hamiltonian vector field

$$i_{X_H}\omega = -dH \quad (\omega \text{ symplectic form on } T^*M)$$

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p} \Rightarrow \gamma \text{ smooth}$$

True definition: for $x \in M$,

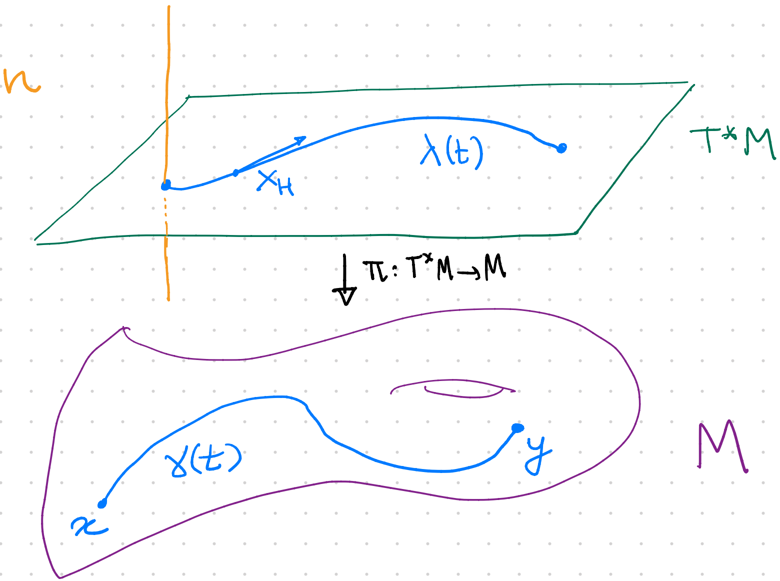
$$H\big|_{T_x^*M}(p) := \frac{1}{2}\max_{v\in D_x\setminus\{0\}}\left\{\frac{p(v)^2}{g_x(v,v)}\right\}$$

Normal ~~minimizers~~ extremals

A normal extremal (NE) is projection $\pi(\lambda)$ of a solution $\lambda(t)$ of the

SUB-RIEMANNIAN HAMILTONIAN

$$H(p,x) = \frac{1}{2}\sum_{i=1}^{k}(p\cdot X_i(x))^2$$



- On fiber T_x^*M

$$H(p,x) = \frac{1}{2}\sum g^{ij}(x)\, p_i p_j$$

however H is degenerate

Dido $H(p,x) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$

$(p,x)\in\mathbb{R}^3\times\mathbb{R}^3$

Grushin $(M=\mathbb{R}^2)$

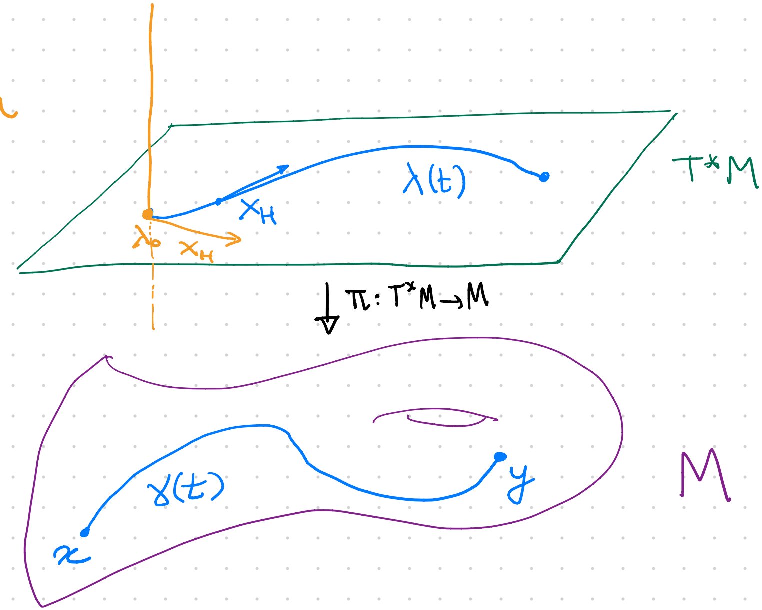
$$H(p,x) = \frac{1}{2}p_1^2 + \frac{1}{2x}p_2^2$$

Normal ~~minimizers~~ extremals

A normal extremal (NE) is projection $\pi(\lambda)$ of a solution $\lambda(t)$ of the

SUB-RIEMANNIAN HAMILTONIAN

$$H(p,x) = \frac{1}{2}\sum_{i=1}^{k} (p \cdot X_i(x))^2$$



- On fiber $T^*_x M$

$$H(p,x) = \frac{1}{2}\sum g^{ij}(x)\, p_i p_j$$

however H is degenerate

- NEs are
  - LOCAL MINIMIZERS
  - determined by initial COVECTOR $\lambda_0 \in T^*M$

Abnormal extremals

An abnormal extremal is a curve γ_u such that

$$\lambda \cdot D_u E_x = 0 \qquad \lambda \in T_y^* M \setminus \{0\} \qquad (\Rightarrow \ u \in \mathrm{Crit}(E_x))$$

In Riemannian case $D_u E_x$ <u>ALWAYS</u> surjective (E_x submersion)

$\Rightarrow$ ABNORMALS CANNOT EXIST

Abnormal extremals

An abnormal extremal is a curve γ_u such that

$$\lambda \cdot D_u E_x = 0 \qquad \lambda \in T_y^* M \setminus \{0\} \qquad (\Rightarrow \; u \in \mathrm{Crit}(E_x))$$

Q: How big is $E_x(\mathrm{Crit}(E_x))$?

Zero measure? Empty interior?

sR-Sard conjecture ???

Morse-Sard theorem

$$\boxed{f \in C^k(N^n, M^m) \Rightarrow \text{ if } k \geq \max\{n-m+1, 1\} \text{ then } f(\mathrm{Crit}(f)) \text{ has measure } 0 \text{ in } M.}$$

Abnormal extremals

An abnormal extremal is a curve γ_u such that

$\lambda \cdot D_u E_x = 0 \qquad \lambda \in T_y^* M \setminus \{0\} \qquad (\Rightarrow u \in \mathrm{Crit}(E_x))$

Q: How big is $E_x(\mathrm{Crit}(E_x))$?
Zero measure? Empty interior?

sR-Sard conjecture ???

A: Mostly open. Recent advances:

- Le Donne, Montgomery, Ottazzi, Pansu, Vittone (2015)
- Belotto, Rifford (2018)
- Belotto, Figalli, Parusiński, Rifford (preprint)

Abnormal extremals (EXAMPLE)

$$M = \mathbb{R}^3, \quad \mathcal{D} = \ker\left(dz - \frac{y^2}{2}dx\right) = \operatorname{span}\Big\{ \overbrace{\partial_y}^{X_1}, \overbrace{\partial_x + \frac{y^2}{2}\partial_z}^{X_2} \Big\}$$

$\Rightarrow$ singular set $Z = \{y = 0\}$ — $\det(X_1, X_2, [X_1, X_2]) = 0$

MARTINET SURFACE

$\Rightarrow$ $\gamma(t) = (\pm t, 0, 0)$ ABNORMAL EXTREMAL

Abnormal extremals (EXAMPLE)

$M=\mathbb{R}^3$, $\mathcal{D}=\ker\left(dz-\frac{y^2}{2}dx\right)=\operatorname{span}\Big\{\underbrace{\partial_y}_{X_1}, \underbrace{\partial_x+\frac{y^2}{2}\partial_z}_{X_2}\Big\}$

$\Rightarrow$ singular set $Z=\{y=0\}$ — $\det(X_1, X_2, [X_1,X_2])=0$

MARTINET SURFACE

$\Rightarrow$ $\gamma(t)=(\pm t,0,0)$ ABNORMAL EXTREMAL

Simplest example.

No abnormal in step 2 Carnot

Plenty of abnormals in quasi-contact and Engels

Turns out that γ

- minimizer regardless of the metric $\leftarrow$ TOPOLOGICAL MINIMIZER
- C^1-isolated among horizontal curves

"As far as I know" these are true for all known examples but it is not known if they hold in general

Abnormal extremals

Very little is known on ABNORMALS

Q: Are ALL sub-Riemannian minimizers smooth?

A: Some partial results:

- Montgomery 1991: $\exists$ strictly abnormal minimizers (smooth)
  $\exists$ non-smooth abnormal extremals (minimizers?)
- Chitour, Jean, Trélat 2006: true if step ≤ 2, generic if rank ≥ 3
- Barilari, Chitour, Jean, Prandi, Sigalotti 2018: all minimizer for rank ≤ 4 and step 2 are C^1
- Hakavuori-Le Donne 2016: length-minimizer do not have corner-type singularities

Recommended Readings (Reprise)

The following were on my desk when I prepared these slides:

- L. Rifford. Sub-Riemannian Geometry & Optimal transport. SpringerBriefs (2014)

- E. Le Donne. Lecture notes on sub-Riemannian geometry. Unpublished, PDF available at enrico.ledonne.googlepages.com (2017)

- A. Agrachev, D. Barilari, U. Boscain. A comprehensive introduction to sub-Riemannian geometry. CUP (2019)

- A. Agrachev. Some open problems. (2013)

PART II

sub-Laplacians & some problems in spectral geometry

RECOMMENDED LITERATURE

Spectral Geometry

the spectral geometry in the clouds seminar, born with the pandemic, did the job for me

REFERENCES tinyurl.com/yxlgf497 (all freely accessible)

OPEN PROBLEMS tinyurl.com/y5cbz8qo

Spectral sub-Riemannian Geometry

[Small & partial account of recent development]

- Boscain, Laurent. Ann. Inst. Fourier (Grenoble) 63 (2013)
- Prandi, Rizzi, Seri. Journal of Spectral Theory 8(4) (2018)
- ——, ——, ——. J. Diff. Geom. 22(4) (2019)
- Chitour, Prandi, Rizzi. Preprint hal-01902740 (2019)
- Colin de Verdière, Hillairet, Trélat. Preprint hal-02535865 (2020) PART OF SERIES OF AT LEAST 4 PAPERS UNDERGOING PREPARATION
- N. Savale. Preprint arXiv:1909.00409
- C. Fermanian-Kammerer, V. Fischer. Preprint arXiv:1910.14486

OUTLINE OF PART II

- SPECTRAL GEOMETRY 101
- SUB-LAPLACIANS & THE PROBLEM OF SUB-RIEMANNIAN VOLUMES
- GRUSHIN SUB-LAPLACIAN, A MOTIVATING EXAMPLE
- THE QUEST FOR ESSENTIAL SELF-ADJOINTNESS
- WEYL'S LAWS & THE REST

SPECTRAL GEOMETRY IN PILLS

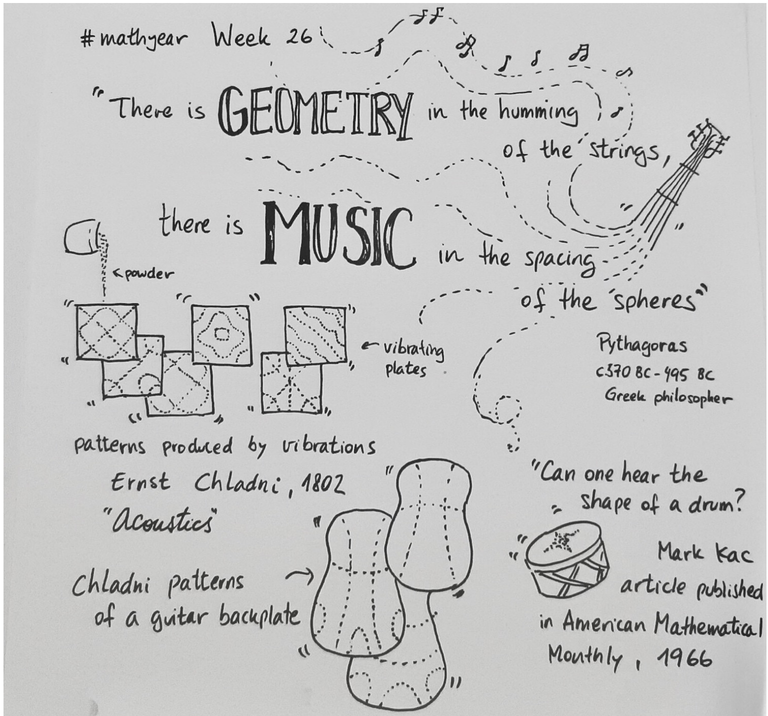


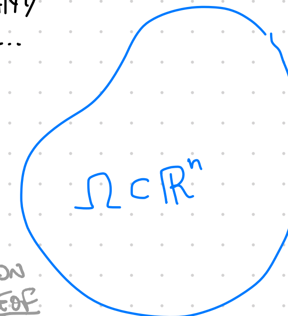
Image courtesy of Constanza Rojas-Molina
http://crojasmolina.com

H. WEYL 1911 AND MANY OTHERS...

DIFFUSION PROCESS IN Ω

$$\begin{cases} -\Delta\psi = \lambda\psi & \text{in} \\ \psi|_{\partial\Omega} = 0 \end{cases}$$

INTERACTION OR LACK THEREOF WITH THE ENVIRONMENT



SPECTRAL GEOMETRY IN PILLS

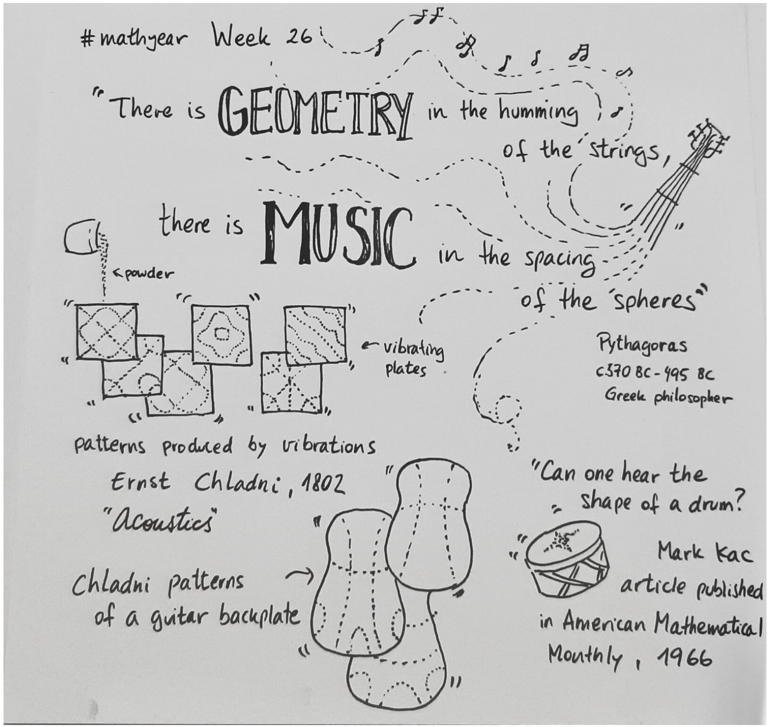
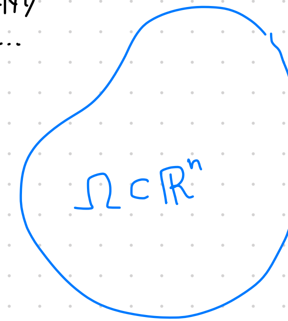


Image courtesy of Constanza Rojas-Molina
http://crojasmolina.com

H. WEYL 1911 AND MANY OTHERS...



$$\begin{cases} -\Delta\psi = \lambda\psi & \text{in} \\ \psi|_{\partial\Omega} = 0 \end{cases}$$

- countably many solutions

$$\lambda_1 > \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \nearrow \infty$$

- $N(\lambda) = \#\{ i \in \mathbb{N} \mid \lambda_i < \lambda \}$

$$= \frac{\omega_n \, \mathrm{vol}(\Omega)}{(2\pi)^n} \lambda^{\frac{n}{2}} - \frac{\mathrm{area}(\partial\Omega)}{4\,(2\pi)^{n-1}} \lambda^{\frac{n-1}{2}} + o\left(\lambda^{\frac{n-1}{2}}\right)$$

SPECTRAL GEOMETRY IN PILLS

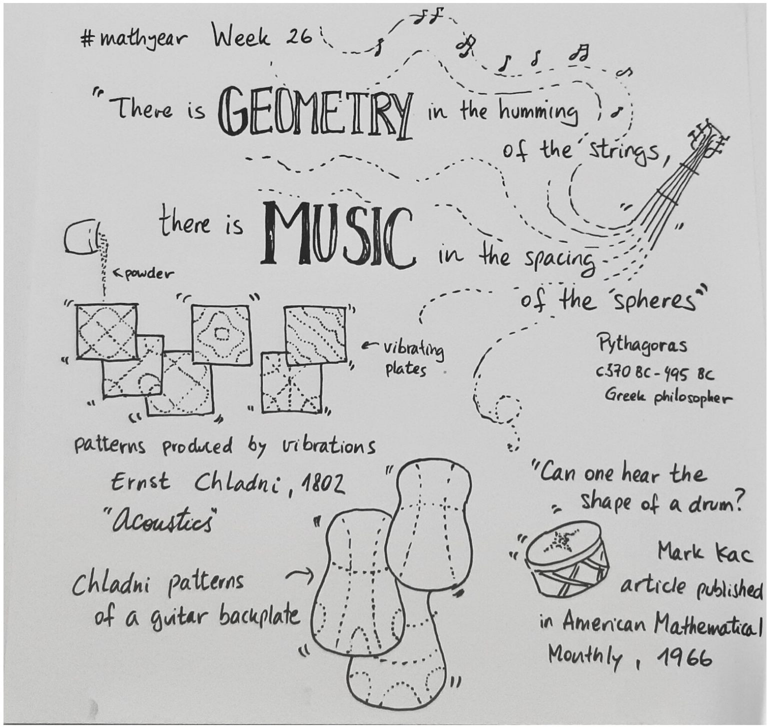
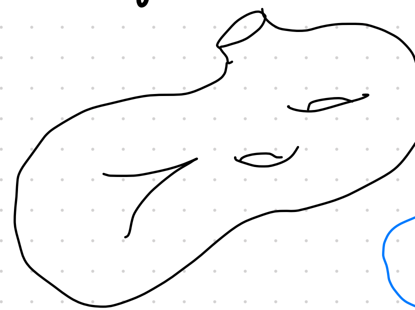


Image courtesy of Constanza Rojas-Molina

(M, μ_g)

GEOMETRY OF A MANIFOLD



SPECTRAL GEOMETRY

SPECTRUM OF LAPLACE-BELTRAMI OPERATOR

$$\Delta_g = \mathrm{div}_{\mu_g} \cdot \nabla_g \quad \text{on } C_0^\infty(M)$$

with boundary conditions on ∂M

SOME QUESTIONS AND SOME ANSWERS

★ M, M' ISOMETRIC $\Rightarrow$ M, M' ISOSPECTRAL. IS THE OPPOSITE TRUE?

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- DIMENSION OF M
- VOLUME OF M
- INTEGRAL OF THE SCALAR CURVATURE S_g OVER M
- (MODULO ADDITIONAL CONDITIONS) LENGTH OF ∂M, GENUS OF M, LENGTH OF CLOSED GEODESICS ON M

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$\boxed{\text{We will come back to this}}$

SOME QUESTIONS AND SOME ANSWERS

* CAN WE FINELY LOCATE THE SPECTRUM USING GEOMETRIC INFORMATION?

SOME QUESTIONS AND SOME ANSWERS

★ CAN WE FINELY LOCATE THE SPECTRUM USING GEOMETRIC INFORMATION?

KIND OF... SPECTRAL INEQUALITIES & MINIMISING MANIFOLDS HAVE BEEN THOROUGHLY DEVELOPED BUT STILL PLENTY OF OPEN QUESTIONS

JUST AN EXAMPLE

M noncompact, $n \geq 2$, M has finite volume.

$$h(M) = \inf \frac{\text{area}(\partial\Omega)}{\text{vol}(\Omega)} \qquad \text{Cheeger's constant}$$

over $\Omega \subset M$ open submanifold, $\partial\Omega$ smooth, ...

$\Rightarrow \lambda_1(\Omega) \geq h^2(\Omega)/4$

OUTLINE OF PART II

- SPECTRAL GEOMETRY 101
- SUB-LAPLACIANS & THE PROBLEM OF SUB-RIEMANNIAN VOLUMES
- GRUSHIN SUB-LAPLACIAN, A MOTIVATING EXAMPLE
- THE QUEST FOR ESSENTIAL SELF-ADJOINTNESS
- WEYL'S LAWS & THE REST

HYPOELLIPTIC OPERATORS IN $\mathbb{R}^n$

Second order linear differential operator on functions on $\mathbb{R}^n$

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

where $a_{ij}, b_i, c \in C^\infty(\mathbb{R}^n, \mathbb{R})$.

L hypoelliptic if, locally, forall distributions u, $Lu \in C^\infty$ implies $u \in C^\infty$

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L hypoelliptic if, locally, for all distributions u, $Lu \in C^\infty$ implies $u \in C^\infty$

$\Rightarrow$ (Hörmander '67) $\underbrace{\sum a_{ij}(x)\xi_i \xi_j}_{\text{Principal Symbol}} \geq 0$ for all $\xi \in \mathbb{R}^n$

N could be $> n$!

$\Rightarrow$ Hörmander-type operators $L = \sum_{i=1}^{N} X_i^2 + X_0 + c$

$X_0, X_1, \dots, X_N$ smooth vector fields on $\mathbb{R}^n$

HYPOELLIPTIC OPERATORS IN $\mathbb{R}^n$

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$\Rightarrow$ Hörmander-type operators $\quad L = \sum_{i=1}^{N} X_i^2 + X_0 + c$

THM (Hörmander '67) $\underbrace{\{X_0, \dots, X_N\} \text{ bracket generating}}_{\mathrm{Lie}_x \{X_0, \dots, X_N\} = \mathbb{R}^n \text{ for all } x \in \mathbb{R}^n} \Rightarrow L$ hypoelliptic

LAPLACE-BELTRAMI OPERATOR ON RIEMANNIAN MANIFOLDS

(M, g) Riemannian manifold of dimension n

$\to$ $\mu_g = \sqrt{\det g}\, dx^1 \wedge \dots \wedge dx^n$ (Riemannian volume)

∇ such that $g(\nabla f, X) = X(f)$ (gradient)

$\to$ $\mathrm{div}_\mu (X) = \sum_{j=1}^{n} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^j} \sqrt{\det g}\, X_k$ (divergence)

$$\boxed{\mathcal{L}_X \mu = (\mathrm{div}\, X)\, \mu}$$

Nice interpretation as rate of expansion of volume as it flows with the vector field

$\to$ Laplace-Beltrami $\Delta := \mathrm{div}_\mu \nabla = \sum_{i,j=1}^{n} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^j} \left(\sqrt{\det g}\, g^{ji} \frac{\partial}{\partial x^i} \right)$

LAPLACE-BELTRAMI OPERATOR ON RIEMANNIAN MANIFOLDS

(M,g) Riemannian manifold of dimension n

→ $\mu_g = \sqrt{\det g}\, dx^1 \wedge \dots \wedge dx^n$ (Riemannian volume)

∇ such that $g(\nabla f, X) = X(f)$ (gradient)

→ $\mathrm{div}_\mu(X)$ such that $\mathcal{L}_X \mu = (\mathrm{div}\, X)\mu$ (divergence)

→ Laplace-Beltrami $\Delta := \mathrm{div}_\mu \nabla$

$\boxed{\text{Riemannian volume}}$: DETERMINED BY THE METRIC → VOLUME OF ORTHONORMAL PARALLELOTOPE IN TM IS 1

$(M, \mathcal{D}, g)$: g defined on $\mathcal{D} \subset TM$ has no canonical extension to TM.

$\boxed{\text{Sub-Riemannian}}$: volume?

SUB-RIEMANNIAN VOLUMES

① (M, d_{SR}) metric space

→ Hausdorff volume $\mathcal{H}^Q$

→ Spherical Hausdorff volume $\mathcal{S}^Q$

Except few cases, it is unknown if these are C^1 (or C^2 like we need)

SUB-RIEMANNIAN VOLUMES

① (M, d_{SR}) metric space → Hausdorff volume $\mathcal{H}^Q$
→ Spherical Hausdorff volume $\mathcal{S}^Q$

② POPP VOLUME $\mathcal{P}$ $\quad [(M,D,g),\ D$ EQUIREGULAR of step $2]$

- ADAPTED FRAME $\{X_1, \dots, X_k, X_{k+1}, \dots, X_n\}$: orthonormal frame of D completed to a frame for T_qM/D_q
- $B_1 := id_{k\times k}$, B_2 $(n-k)$ square matrix defined with brackets of $\{X_{k+1}, \dots, X_n\}$

$\Rightarrow \mathcal{P}$ is volume of parallelotope whose edges are $X_1, \dots, X_n$ as elements of $gr_q(D) = D_q \oplus T_qM/D_q$

looks like a natural extension of Riemannian volume after all!

SUB-RIEMANNIAN VOLUMES

① (M, d_{SR}) metric space $\nearrow$ Hausdorff volume $\mathcal{H}^Q$; $\searrow$ Spherical Hausdorff volume S^Q

② (M, D, g), D EQUIREGULAR of step s, M ORIENTABLE

$\Rightarrow gr_q(D) = D_q \oplus D_q^2/D_q \oplus \cdots \oplus D_q^s/D_q^{s-1}$ can be endowed w. graded Lie algebra

POPP VOLUME

$$\begin{tikzcd}
(M,D) \arrow[r, "P"] \arrow[d, "gr_q"'] & (\Lambda^n T_qM)^* \arrow[d, "\theta_q^*"] \\
gr_q(D) \arrow[r, "\omega"'] & (\Lambda^n gr_q(D))^*
\end{tikzcd}$$

Define inner product on each D_q^i/D_q^{i-1}

$\Rightarrow$ inner product on $gr_q(D)$

$\Rightarrow$ volume form $\omega_q \in \Lambda^n gr_q(D)^*$

$\Rightarrow$ transport ω_q to T_qM by canonical isomorphism $\theta_q : \Lambda^n T_qM \to \Lambda^n gr_q(D)$

$P_q = \omega_q \circ \theta_q = \theta_q^*(\omega_q)$ Popp volume

$\Rightarrow$ use orientability to glue $P \in \Omega^n(M)$

SUB-RIEMANNIAN VOLUMES

① (M, d_{SR}) metric space $\to$ Hausdorff volume $\mathcal{H}^Q$; Spherical Hausdorff volume $\mathcal{S}^Q$ — On Riemannian $(M,\tilde{g})$ these are proportional to $\mu_{\tilde{g}}$

② (M, D, g), D EQUIREGULAR of step s, M ORIENTABLE

POPP VOLUME $\mathcal{P}$

$$\begin{tikzcd}
(M,D) \arrow[r, "\mathcal{P}"] \arrow[d, "gr_q"'] & (\Lambda^n T_q M)^* \arrow[d, "\theta_q^*"] \\
gr_q(D) \arrow[r, "\omega"'] & (\Lambda^n gr_q(D))^*
\end{tikzcd}$$

⓪ $\mathcal{P}$ is smooth

① On Riemannian manifolds $(M,\tilde{g})$, $\mathcal{P} \equiv \mu_{\tilde{g}}$

② Sub-Riemannian isometries preserve Popp volume

③ If $\mathrm{iso}(M,D)$ acts transitively $\Rightarrow$ Popp volume is the only preserved volume

④ $\mathcal{P}$, $\mathcal{H}^Q$ and $\mathcal{S}^Q$ can be related but their relation "is complicated"... [Agrachev, Barilari, Boscain, 2012]

SUB-RIEMANNIAN VOLUMES

Q: Can you define a canonical smooth volume form from a sub-Riemannian structure (M, D, g)?

① (M, d_{SR}) metric space $\nearrow$ Hausdorff volume $\mathcal{H}^Q$
$\searrow$ Spherical Hausdorff volume $\mathcal{S}^Q$

② (M, D, g) equiregular $\longrightarrow$ Popp volume (if nilpotent approx at different pts are isometric, otherwise ∞-many)

③ 3D contact $\longrightarrow$ Weyl measure W_Δ
[Y. Colin de Verdière, Hillaret, Trélat, 2018]

④ More?

Non-equiregular case or rank-varying case is fully open!

THE SUBLAPLACIAN

From now on: • $(M, D = \mathrm{span}\{X_1, \dots, X_k\}, g)$ sub-Riemannian manifold

• D bracket-generating

• $\mu \in \Omega^n(M)$ smooth volume on M

$\Rightarrow$ HORIZONTAL GRADIENT $\quad \langle \nabla_H f, X \rangle_g = X(f) \qquad X \in \Gamma(D), \quad f \in C^\infty(M)$

$\Rightarrow$ DIVERGENCE $\quad \mathcal{L}_X \mu = \mathrm{div}_\mu(X)\mu \qquad X \in \mathrm{Vec}(M)$

The sub-laplacian (associated with μ) is $\boxed{\Delta_\mu f = \mathrm{div}_\mu(\nabla_H f)}$

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The sub-laplacian (associated with μ) is

$$\Delta_\mu f = \mathrm{div}_\mu(\nabla_H f) = \sum_{i=1}^k \mathrm{div}_\mu(X_i(f) X_i)$$

$$= \sum_{i=1}^k X_i(X_i(f)) + \mathrm{div}_\mu(X_i) X_i(f)$$

$$\nabla_H f = \sum_{i=1}^k X_i(f) X_i$$

$$\mathrm{div}_\mu(fX) = X(f) + f\,\mathrm{div}_\mu X$$

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$$= \sum_{i=1}^k X_i(X_i(f)) + \mathrm{div}_\mu(X_i) X_i(f)$$

$$\mathrm{div}_\mu(fX) = X(f) + f \, \mathrm{div}_\mu X$$

$$\Rightarrow \boxed{\Delta_\mu = \sum_{i=1}^k X_i^2 + X_0, \qquad X_0 = \sum_{i=1}^k \mathrm{div}_\mu(X_i) X_i}$$

Hörmander type operator

THE SUBLAPLACIAN II

$$\Delta = \underbrace{\sum_{i=1}^{k} X_i^2}_{\text{DOES NOT DEPEND ON } \mu} + X_0, \qquad X_0 = \sum_{i=1}^{k} \operatorname{div}_\mu(X_i) X_i$$

Properties

① $\int_M f \Delta g \, \mu = - \int_M \langle \nabla f, \nabla g \rangle \mu \qquad \forall f, g \in C_0^\infty(M)$

$\Rightarrow \Delta = - \sum_{i=1}^{k} X_i^* X_i$ is SYMMETRIC, NEGATIVE on $C_0^\infty(M)$

② If (M, d_{SR}) complete $\Rightarrow$
- Δ is essentially self-adjoint on $L^2(M, \mu)$ w. domain $C_0^\infty(M)$

Link w. Stratonovich SDEs & Malliavin calculus $\longrightarrow$
- the heat operator $e^{t\Delta}$ has a positive C^∞ kernel and is contractive as semigroup

SUB-LAPLACIANS (EXAMPLES)

- (M, TM, g) Riemannian $\Rightarrow$ Δ_{μ_g} Laplace-Beltrami operator

- Dido (Heisenberg) with Haar measure from left-invariant group action: $\mu = dx \wedge dy \wedge dz$

$$\Delta = X_1^2 + X_2^2$$
$$= \left(\frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}\right)^2 + \left(\frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}\right)^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{4}(x^2+y^2)\frac{\partial^2}{\partial z^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial z}$$

(modulo "i"s and "$\hbar$"s this is a magnetic Schrödinger operator on $L^2(\mathbb{R}^3)$ for constant field in z direction)

- Grushin on $\mathbb{R}^2$ with Grushin metric $g = dx^2 + \frac{1}{x^2} dy^2$ (ALMOST-RIEMANNIAN NON-EQUIREGULAR)

$$\Rightarrow \mu_g = \frac{1}{|x|} dx dy \quad \text{and} \quad X_1 = \frac{\partial}{\partial x}, \; X_2 = x\frac{\partial}{\partial y}$$

$$\Rightarrow \Delta = \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2} - \frac{1}{x}\frac{\partial}{\partial x}$$

Outline of Part II

- Spectral geometry 101
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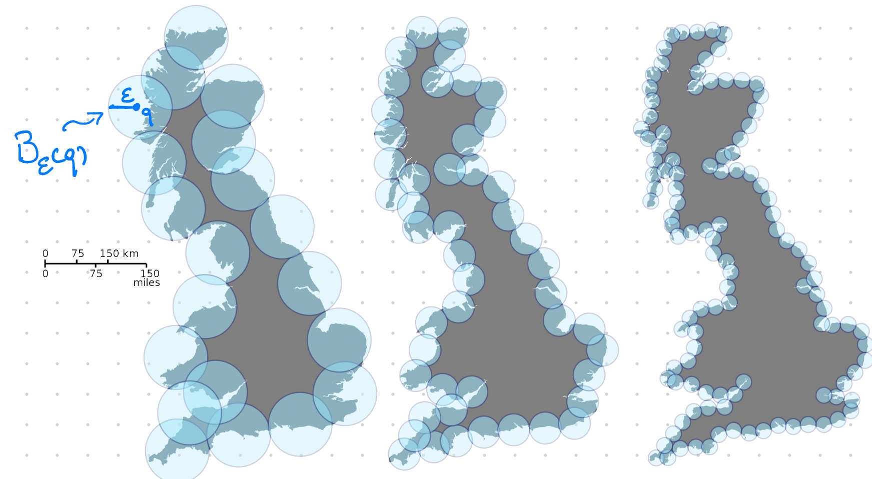
Hausdorff dimension (TECHNICAL INTERMISSION)



[Wikimedia Foundation]

(X, d) metric space

$\Rightarrow$ Hausdorff dimension

$$Q(q) = \lim_{\varepsilon \to 0} \frac{\ln(\mathrm{vol}\, B_\varepsilon(q))}{\ln \varepsilon}$$

BALL-BOX THEOREM

In the case of a equiregular subRiemannian structure (M^n, D, g)

$$= \sum_{j=1}^{s} j\left(\dim D_q^j - \dim D_q^{j-1}\right) \begin{cases} > n & \text{in sub-Riemannian case} \\ = n & \text{in Riemannian case} \end{cases}$$

GRUSHIN LAPLACIAN: A MOTIVATING EXAMPLE

GRUSHIN CYLINDER $M = \mathbb{R}_x \times \mathbb{S}^1_\theta$ with $X_1 = \partial_x$, $X_2 = x\partial_\theta$ and $g = dx^2 + \frac{d\theta^2}{x^2}$

$\Rightarrow$ geodesic flow is flow of $H = p_x^2 + x^2 p_\theta^2$

GRUSHIN LAPLACIAN: A MOTIVATING EXAMPLE

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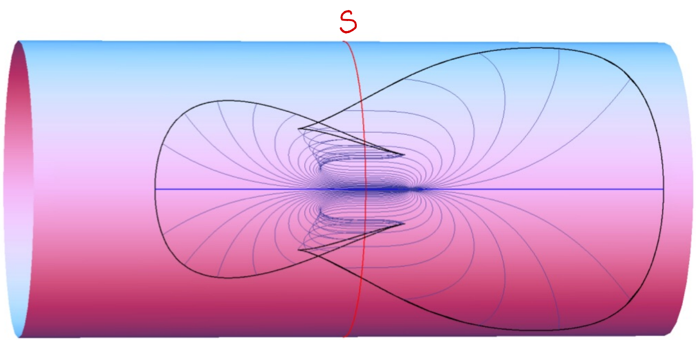
$\Rightarrow$ geodesic flow is flow of $H = p_x^2 + x^2 p_\theta^2$

$\mu_g = \frac{1}{|x|}\, dx\, d\theta$

Riemannian in $\mathcal{R} = M \setminus S$

Singular set $S := \{x = 0\}$ where $\operatorname{rank} D = 1$



Geodesic front of length L emanating from $x = 0$

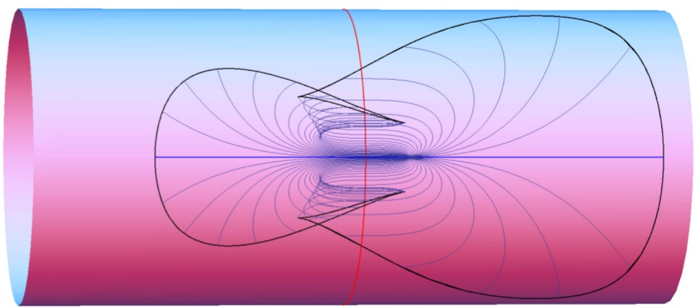
$\leftarrow$ NOT GEODESICALLY COMPLETE ON $\mathcal{R}$!

GRUSHIN LAPLACIAN: A MOTIVATING EXAMPLE

GRUSHIN CYLINDER $M = \mathbb{R}_x \times \mathbb{S}^1_\theta$ with $X_1 = \partial_x$, $X_2 = x\partial_\theta$ and $g = dx^2 + \frac{d\theta^2}{x^2}$

$\Rightarrow$ geodesic flow is flow of $H = p_x^2 + x^2 p_\theta^2$, $\mu_g = \frac{1}{|x|}\, dx\, d\theta$

$$\Delta = \partial_x^2 - \frac{1}{x}\partial_x + x^2 \partial_\theta^2 \quad \text{on} \quad L^2\left(M, \frac{1}{|x|} dx\, d\theta\right)$$



Geodesic front of length L emanating from $x=0$

$$\boxed{\begin{array}{l} \mathrm{Vol}(M) \text{ is infinite} \\ K = -\dfrac{2}{x^2} \xrightarrow[x \to 0]{} -\infty \end{array}}$$

DOES THE WEYL'S LAW MAKE ANY SENSE?

[Agrachev, Barilari, Rizzi, 2018]

Note: due to the absence of canonical affine connection, making sense of curvatures in sub-Riemannian setting is active area of research!

GRUSHIN LAPLACIAN: A MOTIVATING EXAMPLE

GRUSHIN CYLINDER $M = \mathbb{R}_x \times \mathbb{S}^1_\theta$ with $X_1 = \partial_x$, $X_2 = x\partial_\theta$ and $g = dx^2 + \frac{d\theta^2}{x^2}$

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THEOREM (BOSCAIN-PRANDI-S, 2015)

On each half of $M \setminus \{x=0\}$ the operator Δ is essentially self-adjoint on C_0^∞ functions and its spectrum is $[0,\infty)$ continuous with embedded eigenvalues $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$

Moreover

$$N(\lambda) = \frac{\lambda}{2}\log\lambda + \left(\gamma - \log 2 - \frac{1}{2}\right)\lambda + O(\sqrt{\lambda})$$

[In fact of $\lambda^{\frac{131}{416}+\varepsilon}$ $\forall \varepsilon > 0$

$\lambda^{+Q/2}$

Same computation in Martinet case gives $N(\lambda) \sim c\lambda^2 \log\lambda$

GRUSHIN LAPLACIAN: A MOTIVATING EXAMPLE

Theorem (Boscain-Prandi-S, 2015)

On each half of $M \setminus \{x=0\}$ the operator Δ_g is essentially self-adjoint on C_0^∞ functions and its spectrum is $[0,\infty)$ continuous with embedded eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$

Moreover

$$N(\lambda) = \frac{\lambda}{2}\log\lambda + \left(\gamma - \log 2 - \frac{1}{2}\right)\lambda + O(\sqrt{\lambda})$$

Q: what is this in general? what does it measure?

Q: how general is this phenomenon?

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SELF-ADJOINTNESS OF DIFFERENTIAL OPERATORS ON MANIFOLDS

CONJECTURE [Colin de Verdière, Le Bihan, 2020]

X closed smooth manifold with smooth density μ.
P formally self-adjoint differential operator on $C^\infty(X)$.

The Hamiltonian flow of symbol p of P is complete

IF AND ONLY IF

P is essentially self-adjoint

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<u>IF AND ONLY IF</u>

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Some part of this is well known:

- (X, g) Riemannian and complete $\Rightarrow$ Δ essentially self-adjoint (ESA)
- (X, D, g) equiregular sub-Riemannian and complete $\Rightarrow$ Δ_p ESA

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P formally self-adjoint differential operator on $C^\infty(X)$.

The Hamiltonian flow of symbol p of P is complete

IF AND ONLY IF

P is essentially self-adjoint

True for :
- differential operators of deg 2 on S^1 + minor assumptions on principal symbol
- differential operators of deg 1
- generic Lorentzian Laplacians on surfaces

False for almost-Riemannian manifolds (like our Grushin example)

SELF-ADJOINTNESS FOR RANK-VARYING SUB-LAPLACIANS

- (M, D, g) complete sub-Riemannian manifold
- $S \subset M$ smooth embedded compact hypersurface with NO CHARACTERISTIC POINTS
- ω smooth measure on $\boxed{M \setminus S}$ — d_{SR} is Riemannian on R

Assume that for some $\varepsilon > 0$, $\exists\, \kappa \geq 0$ such that [Prandi, Rizzi, Seri 2019]

$$\left(\frac{\Delta_\omega \delta}{2}\right)^2 + \left(\frac{\Delta_\omega \delta}{2}\right)' \geq \frac{3}{4\delta^2} - \frac{\kappa}{\delta} \qquad \text{for } 0 < \delta \leq \varepsilon \qquad (\star)$$

$\boxed{\delta := d(S, \cdot)}$

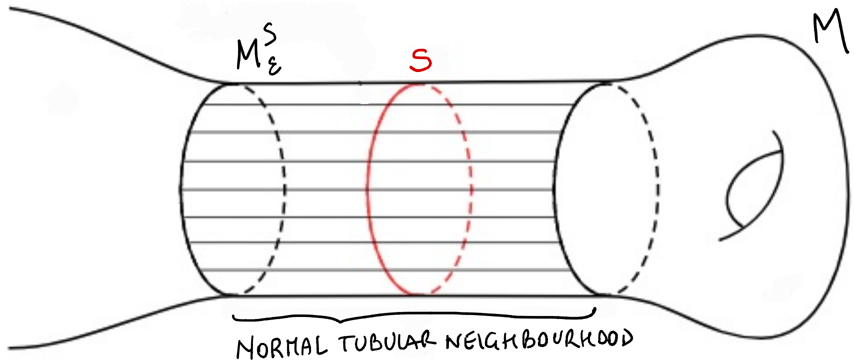
Directional derivative in direction $\nabla\delta$

$\Rightarrow$ Δ_ω with domain $C_0^\infty(M \setminus S)$ is essentially self-adjoint on $L^2(M \setminus S, \omega)$ or any of the connected components of $M \setminus S$.

If $M \setminus S$ relatively compact $\Rightarrow$ Δ_ω has compact resolvent $\leftarrow$ We can think of its Weyl law.

SELF-ADJOINTNESS FOR RANK-VARYING SUB-LAPLACIANS

CRUCIAL INGREDIENT OF THE PROOF



$M^S_\varepsilon \simeq (0,\varepsilon) \times X_\varepsilon \sqcup (-\varepsilon, 0) \times X_\varepsilon$

$X_\varepsilon = \{\delta(q) = \varepsilon\}$

$\oplus$ ON M^S_ε, $\mu \sim \delta^{-\alpha} \times$ SMOOTH MEASURE

Riemannian volume from the Riemannian metric for d_{SR}

for some $\alpha \geq 1$

In the case of the picture & of Grushin one can take $X_\varepsilon = S$.

SELF-ADJOINTNESS FOR RANK-VARYING SUB-LAPLACIANS

PROOF

Weak Hardy inequality

REQUIRES PRECISE AND DEEP UNDERSTANDING OF GEODESIC FLOW

Under ⊛, there exist $\eta \leq 1/k$ & $c \in \mathbb{R}$ such that

$$\int_M |\nabla u|^2 d\omega \geq \int_{M_\eta^S} \left(\frac{1}{\delta^2} - \frac{k}{\delta}\right)|u|^2 d\omega + c\|u\|^2 \qquad \odot$$

for u in a suitable Sobolev space.

$\Rightarrow \Delta_\omega$ semibounded on $C_c^\infty(M)$

[Prandi, Rizzi, Seri, 2018]
[Franceschi, Prandi, Rizzi, 2019]

$\longrightarrow$

Agmon-type estimate

Under ⊛, if $\odot$ holds, then $\forall E < c$ the only solution of $\Delta_\omega^* \psi = E\psi$ is $\psi \equiv 0$.

ESSENTIALLY PREVENTS WEAK SOLUTIONS TO BE SUPPORTED NEAR S.

$\downarrow$

Self adjointness criterion [Reed, Simon, X.1]

Since Δ_ω semibounded. Δ_ω is ESA if and only if $\exists E < 0$ such that the only solution of $\Delta_\omega^* \psi = E\psi$ is $\psi \equiv 0$.

SELF-ADJOINTNESS FOR RANK-VARYING SUB-LAPLACIANS

GENERAL CASE IS STILL WIDELY OPEN AND REQUIRES COMPLETELY NEW IDEAS!

<u>EXAMPLE</u> $M = \mathbb{R}^2$, $X_1 = \frac{\partial}{\partial x}$, $X_2 = x(x^{2\ell} + y^2)\frac{\partial}{\partial y}$

$\Rightarrow S = \{x = 0\}$, $\delta = |x|$ and $\mu_g = \frac{1}{|x|(x^{2\ell} + y^2)}\,dx\,dy$

- $\ell = 1$ special : one can explicitly compute Δ_μ is ESA
- $\ell > 1$ unknown CONJECTURE is that it is ESA

PROBLEM: $\mu_g \not\approx \delta^{-\alpha} \times$ smooth measure close to S

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<u>EXAMPLE</u> $M=\mathbb{R}^2$, $X_1=\frac{\partial}{\partial x}$, $X_2 = \phi(x,y)\frac{\partial}{\partial y}$, $\phi(x,y) = y-x^2$

$\Rightarrow S = \{\phi(x,y)=0\}$ and 0 tangency point $(D_0 /\!/ T_0 S)$. Here $\delta \approx |\sqrt{y}-|x||$.

Here $\mu_g = \frac{1}{|\phi(x,y)|}\,dx\,dy$ and $\Delta = \partial_x^2 + \phi^2\partial_y^2 + \phi\partial_y + \frac{\partial_x\phi}{\phi}\partial_x$ is symmetric

but its self-adjointness is unknown.

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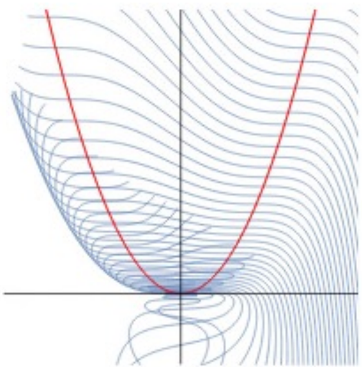
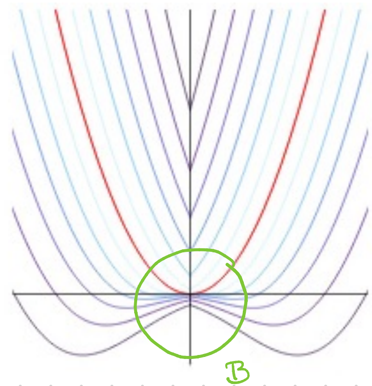
$\Rightarrow S=\{\phi(x,y)=0\}$ and 0 tangency point $(D_0 /\!/ T_0 S)$

GEODESICS STARTING AT $x>0$



LEVELSETS OF δ



$\delta \approx |\sqrt{y}-|x||$

numerically but not rigorously known away from $(0,0)$

Hardy-type results can be shown away from B, currently not in B.

HAVING EXPLICIT FORMULAS IS NOT ENOUGH IT SEEMS

OUTLINE OF PART II

- SPECTRAL GEOMETRY 101
- SUB-LAPLACIANS & THE PROBLEM OF SUB-RIEMANNIAN VOLUMES
- GRUSHIN SUB-LAPLACIAN, A MOTIVATING EXAMPLE
- THE QUEST FOR ESSENTIAL SELF-ADJOINTNESS
- WEYL'S LAWS & THE REST

SUB-RIEMANNIAN WEYL'S LAW

Mostly open, same for quantum ergodicity!

- EQUIREGULAR CASE: $N(\lambda) \sim \dfrac{\lambda^{Q/2}}{\Gamma(Q/2+1)} \int_M \mathcal{Q}_0$ [Métivier, 1976]

- GRUSHIN, MARTINET 2d [Boscain, Prandi, Seri, 2015]

- 3D CLOSED MANIFOLDS W. ORIENTED CONTACT DISTRIBUTION

$$N(\lambda) \sim \frac{\mathcal{P}(M)}{32} \lambda^2$$

[Colin de Verdière, Hillairet, Trélat, 2018]

⊕ Quantum Ergodicity for ergodic Reeb flow

- 4D QUASI-CONTACT

$$N(\lambda) \sim P(M) \lambda^{5/2}$$

[Savale, 2019]

⊕ contribution to spectral support by abnormal geodesics

- SINGULAR SR MANIFOLDS

$$\exists c>0 \text{ st } \frac{1}{c} \leq \frac{N(\lambda)}{\lambda^{Q/2}} \leq c$$

[Chitour, Prandi, Rizzi, 2019]

SUB-RIEMANNIAN SPECTRAL GEOMETRY

We have just seen a small panoramic of SOME open questions.
The field is extremely young, there is much more to do than what you saw today

- 3D CONTACT: are Reeb periods spectral invariants?
- 5D CONTACT: resonance of two harmonic oscillators ⇒ cannot really use normal form approach
- QUANTUM ERGODICITY: mostly open even in low dimensional examples (Martinet)
- CONTROLLABILITY & OBSERVABILITY of subelliptic heat & wave equations: also mostly open
- ...


"That's all Folks!"
Thanks!